Logic 2: Modal Logics – Week 1

Propositional Logic: A Recap

- **Vocabulary of Propositional Logic** (PL):
  - Sentence Letters: p, q, r ...
  - Logical Connectives: $\neg$, $\vee$
  - Parentheses: (,)

Since there is an infinite number of sentence letters, if we run out of letters, we will simply use superscripts: p^1, p^2, p^3 ... p^n.

- **Syntax of PL**:

(a) Every sentence letter is a well-formed formula (wff).
(b) If ϕ and ψ are wffs, then $(\phi \vee \psi)$ and $\neg\phi$ are also wffs.
(c) Nothing else is a wff.

- **Additional Connectives**: We define the additional standard connectives in terms of '$\neg$' and '$\vee$':
  - $\phi \wedge \psi =_{def} \neg(\neg\phi \vee \neg\psi)$
  - $\phi \to \psi =_{def} \neg\phi \vee \psi$
  - $\phi \leftrightarrow \psi =_{def} (\phi \to \psi) \wedge (\psi \to \phi)$

Keeping the primitive connectives to a minimum is beneficial if various meta-logical results such as *soundness* and *completeness* are to be proved. However, since we are not going to do this in this class, this is purely for illustrative purposes.

- **Semantics of PL**:

In PL, meaning is explicated in terms of truth. In general, a (total) assignment of truth values to sentence letters is referred to as an *interpretation* (or a *model*).

DEFINITION OF AN INTERPRETATION:
A PL-interpretation is a total function $\mathcal{I}$ that assigns to each sentence letter the value 1 or 0.

An *interpretation* is also referred to as a *model*. That is, a model for PL consists of a domain of truth values and, where S is the set of all propositional variables, a (total) function $\mathcal{I}$: $S \mapsto \{0,1\}$. This function is standardly called the *interpretation* function

- So, interpretations may look as follows:

$\mathcal{I}(p) = 1$	$\mathcal{I}'(p) = 0$	$\mathcal{I}''(p) = 0$
$\mathcal{I}(q) = 0$	$\mathcal{I}'(q) = 1$	$\mathcal{I}''(q) = 1$
$\mathcal{I}(r) = 1$	$\mathcal{I}'(r) = 1$	$\mathcal{I}''(r) = 0$
...	...	...
$\mathcal{I}(p^n) = 0$	$\mathcal{I}'(p^n) = 0$	$\mathcal{I}''(p^n) = 0$

- The connectives in PL are truth functional, cf. the truth tables below, so given an interpretation $\mathcal{I}$, the truth value of any complex wff is determined by the truth tables for the relevant connectives.

$\neg$	
1	0
0	1

$\vee$	1	0
1	1	0
0	0	0

- However, for full generality, we include a formal definition of a function, a *valuation* function $\mathcal{V}$, that assigns truth values to complex formulas as a function of the truth values of the sentence letters, viz. as a function of the values determined by the interpretation function $\mathcal{I}$.

  - DEFINITION OF A VALUATION
    For any PL-interpretation $\mathcal{I}$, the valuation function $\mathcal{V}$ is the function that assigns to each wff either 1 or 0, and which is such that for any sentence letter α and wff ϕ and ψ:
    - $\mathcal{V}_{\mathcal{I}}(\alpha) = \mathcal{I}(\alpha)$
    - $\mathcal{V}_{\mathcal{I}}(\neg\phi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\phi) = 0$
    - $\mathcal{V}_{\mathcal{I}}(\phi \vee \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\phi) = 1$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$
  - And accordingly ...
    - $\mathcal{V}_{\mathcal{I}}(\phi \wedge \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\phi) = 1$ and $\mathcal{V}_{\mathcal{I}}(\psi) = 1$
    - $\mathcal{V}_{\mathcal{I}}(\phi \rightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\phi) = 0$ or $\mathcal{V}_{\mathcal{I}}(\psi) = 1$
    - $\mathcal{V}_{\mathcal{I}}(\phi \leftrightarrow \psi) = 1$ iff $\mathcal{V}_{\mathcal{I}}(\phi) = \mathcal{V}_{\mathcal{I}}(\psi)$

- Next, we define *validity* in PL:

  DEFINITION OF VALIDITY IN PL
  A wff ϕ is PL-valid iff for every PL-interpretation $\mathcal{I}$, $\mathcal{V}_{\mathcal{I}}(\phi) = 1$.

That a wff ϕ is PL-valid is typically written as '$\vDash_{\text{PL}} \phi$'. The PL subscript is ignored when it is obvious the language in question is PL.

- And logical consequence:

  LOGICAL CONSEQUENCE IN PL
  A wff ϕ is a PL-logical consequence of a set of wffs Γ iff for every PL-interpretation $\mathcal{I}$, if $\mathcal{V}_{\mathcal{I}}(\gamma) = 1$ for each γ such that $\gamma \in \Gamma$, then $\mathcal{V}_{\mathcal{I}}(\phi) = 1$.

That a wff ϕ is a logical consequence of a set of wffs Γ is typically written as '$\Gamma \vDash_{\text{PL}} \phi$'. Again, the PL subscript is ignored when it is obvious the language in question is PL.

- That is, a wff ϕ is a PL-logical consequence of another set of wffs iff whenever the set of wffs are true, ϕ is true too. In other words, on this (semantic) conception of logical consequence, logical consequence is truth preservation (while keeping the meaning of the connectives fixed).

- Note that if ϕ is PL-*valid*, it is a PL- logical consequence of any Γ including when $\Gamma = \varnothing$, i.e. when Γ is empty.

- **Establishing Validity: Semantic Trees/Tableaux**
  There are several methods for checking whether a formula is valid. For example, one method is truth tables. In a truth table it is easily checked whether a formula is true under every interpretation (i.e. in every row for the main connective). To illustrate, consider the wff in (1).

  (1) $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$

- This wff is valid and this can be demonstrated by constructing a truth table:

p	q	$(p \to q)$	$\to$	$(\neg q \to \neg p)$
1	1	1 **1** 1	**1**	0 **1** 0
1	0	1 **0** 0	**1**	1 **0** 0
0	1	0 **1** 1	**1**	0 **1** 1
0	0	0 **1** 0	**1**	1 **1** 1

If we were being absolutely precise, the formulas containing negation in the last column should be values both for the sentence itself (the atomic formula) and for the complex formula including the negation.

- However, this method becomes cumbersome quite quickly, especially as the number of atomic formulas increases.
- A less cumbersome method is so-called semantic trees or semantic tableaux. The purpose of constructing a semantic tree is to search for a counterexample. If the wff is valid, the search is going to lead to a contradiction. Here is an example of the process:
  - One starts by *Negating the Target Formula* (NTF).

1. $\neg\big((p \to q) \to (\neg q \to \neg p)\big)$ NTF $\Longleftarrow$

In the column on the right, we indicate the line that justifies the extension of the tree.

  - Next, the NTF has to be "unpacked" into its logical consequences. Since the NTF in line 1. is a false conditional, then given the semantics for '¬' and →', this means that the antecedent of the conditional must be true and that the consequent must be false. So, we unpack the wff by adding to the tree as follows:

2. $(p \to q)$ l.1
3. $\neg(\neg q \to \neg p)$ l.1

  - Next, we need to unpack the formulas in lines 2 and 3. In line 3 we have another false conditional, so we unpack this using the same procedure, viz. antecedent true, consequent false.

4. $\neg q$ l.3
5. $\neg\neg p$ l.3

  - Given the semantics for '¬', we unpack 5. as follows:

Since: $\neg\neg p \vDash_{\text{PL}} p$

6. p l.5

  - Now, we need to unpack the formula in line 2. There are two options sufficient for this conditional to be true: either the antecedent is false *or* the consequent is true. Since we do not know which, we must consider both options. As a result, the tree now *branches* into the two relevant options, namely p and $\neg q$.

7.	$\neg p$ $\quad\quad$ q		

- But notice that each branch (starting from the bottom and working your way to the top) will now contain either p and $\neg p$ or q and $\neg q$. In other words, by working our way up the branch, we can generate a contradiction. Whenever a branch leads to a contradiction, we say that the branch *closes*. If all branches of a tree close, this shows that when the target formula is negated, it inevitably leads to a contradiction. So, the negation of the negation of the target formula must be true. And so, the target formula is valid.
- The full derivation tree looks as follows:

1.	$\neg((p \to q) \to (\neg q \to \neg p))$		NTF
2.	$p \to q$	√	l.1
3.	$\neg(\neg q \to \neg p)$	√	l.1
4.	$\neg q$		l.3
5.	$\neg\neg p$	√	l.3
6.	p		l.4
7.	$\neg p$ $\quad\quad$ q		l.2
	× $\quad\quad$ ×		

In general, you should include line numbers on the left and an indication of the line that justifies the move on the right. The checkmarks are only meant for bookkeeping — to be added as the relevant formulas are unpacked.

- By contrast, if a target formula is invalid, the tree will not close. For illustration, consider the formula below:

(2) $(p \to q) \to (p \to r)$

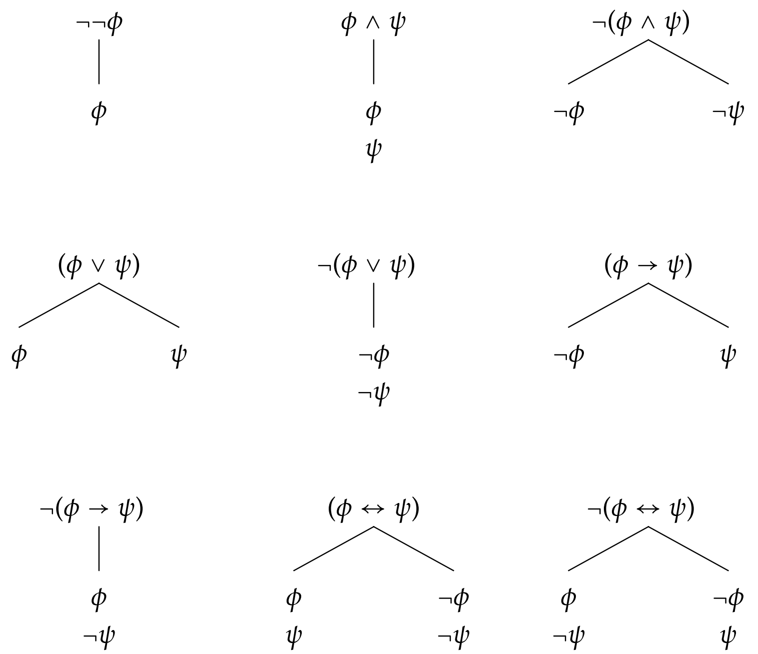
- The semantic tree for this wff looks as follows:

1.	$\neg((p \to q) \to (p \to r))$		NTF
2.	$p \to q$	√	l.1
3.	$\neg(p \to r)$	√	l.1
4.	p		l.3
5.	$\neg r$		l.3
6.	$\neg p$ $\quad\quad$ q		l.2
	× $\quad\quad$ ↑		

- The full set of contruction rules for semantic trees is given below.

These rules can, of course, be mechanically applied, but it is important that you take the time to *understand* why the rules are as they are.

Construction Rules for Semantic Trees in PL



EXERCISE: Show that the following wffs are valid by constructing semantic trees:

(a) $\neg(p \wedge q) \to (\neg p \vee \neg q)$
(b) $\neg(p \to (q \to p)) \to \neg(r \to q)$
(c) $(r \to (r \to (q \wedge p))) \to (q \to (p \vee q))$

- **Invalid Formulas**
  - If a wff is valid, it is true under every interpretation, so a wff will be invalid only if there is an interpretation (i.e. an assignment of truth values to the relevant atomic formulas) such that the wff is false. Hence, to demonstrate that a formula is false, one must demonstrate that such an interpretation exists.

This is also standardly referred to as constructing a "countermodel".

  - Consider again the example in (2) above. This is invalid, because there is an interpretation where the sentence is false, namely the following:
    - $\mathcal{I}(p) = 1$ · $\mathcal{I}(q) = 1$ · $\mathcal{I}(r) = 0$
  - Given this interpretation, we get the following valuations:
    - $\mathcal{V}_{\mathcal{I}}(p \to q) = 1$
    - $\mathcal{V}_{\mathcal{I}}(p \to r) = 0$
  - And hence,
    - $\mathcal{V}_{\mathcal{I}}((p \to q) \to (p \to r)) = 0$
  - Notice that while a specific interpretation must be determined in order to establish that a wff is invalid, such an interpretation can be easily established by using a semantic tree. In particular, by simply working your way up an open branch of the tree, the relevant interpretation is easily determined, cf. above.

Modal Propositional Logic

- Modal Propositional Logic (MPL) is an extension of propositional (PL) that allows us to characterize the validity and invalidity of arguments with *modal* premises or conclusions.
- Specifically, modal logic is intended to help account for the validity of arguments that involve statements such as (3)–(7).

Later, we will use the modal framework to also consider statements such as (a)-(e) below.

(a) It will be that p.
(b) a knows that p.
(c) a believes that p.
(d) It is obligatory that p.
(e) It is permitted that p.

(3) It is necessary that ϕ.
(4) It is must be that ϕ.
(5) It is possible that ϕ.
(6) It might be that ϕ.
(7) It may be that ϕ.

- In modal propositional logic (MPL), the logical vocabulary of propositional logic (PL) is enriched with two monadic operators, namely '$\Box$' and '$\Diamond$', which combine with any atomic or complex PL formula.

That is, '$\Box$' and '$\Diamond$' are operators that can be combined with a wff, i.e. '$\Box\phi$' and '$\Diamond\phi$'. When a modal operator is combined with a sentence ϕ, we refer to ϕ as the *embedded* sentence or the *embedded* formula.

- The meaning of '$\Box$' is roughly 'it is necessary that' (or 'it must be that' or 'necessarily') and the meaning of $\Diamond$ is roughly 'it is possible that' (or 'it might be that' or 'possibly'). Moreover, the meaning of '$\Box$' and '$\Diamond$' is explicated in terms of *possible worlds* (more on this in a bit).
- Simplifying somewhat, the meaning of '$\Box\phi$' and '$\Diamond\phi$' is going to be the following:

'$\Box\phi$' = 1 iff 'ϕ' = 1 in *every* possible world.

'$\Diamond\phi$' = 1 iff 'ϕ' = 1 in *some* possible world.

- So '$\Box$' and '$\Diamond$' are essentially monadic operators that function as *quantifiers* over possible worlds.

Specifically, you can think of $\Box$ as a universal quantifier over possible worlds and $\Diamond$ as an existential quantifier (more on this later).

- When '$\Box$' and '$\Diamond$' are added to PL, we will be able to translate various modal sentences in the following way:

(8) It's necessary that water is H_2O $\rightsquigarrow \Box p$
(9) If water is H_2O, then bread must contain H_2O. $\rightsquigarrow \Box(p \rightarrow q)$
(10) Possibly, water is not H_2O, but necessarily, it is. $\rightsquigarrow \Diamond\neg p \wedge \Box p$
(11) Necessarily, water is either H_2O or it might be *XYZ*. $\rightsquigarrow \Box(p \vee \Diamond r)$

- In order for this to make sense, we need to make various changes to our initial PL system. Most importantly, we need to amend the models to contain possible worlds and make formulas true or false relative to such worlds.
- But before engaging in this project, it is worth considering why going to all this trouble is really needed. For example, why couldn't we just stick with PL and introduce $\Box$ and $\Diamond$ as truth functional

Remember, the truth of a formula containing a truth functional connective is determined entirely by the truth values of the arguments of the connective.

connectives (on the model of negation)?

- **The Problem with a Truth-Functional Analysis of □ and ◇:** A connective is truth functional under the following condition: Whenever it is combined with one or more atomic sentences to form a new (complex) sentence ϕ, the truth value of ϕ is a function of the truth values of its component sentence(s).

- It is now easily demonstrated that a truth functional analysis of modal expressions is inadequate. Consider the sentences below:

(12) Brian Rabern is in Quebec.
(13) It's possible that Brian Rabern is in Quebec.
(14) 4 is a prime number.
(15) It's possible that 4 is a prime number.

- Let's start by looking at '◇': What should the truth table for ◇ look like?

ϕ	1	0
$\Diamond\phi$	?	?

  - If (12) is true, i.e. if it is true that Brian Rabern is in Quebec, then (13) is intuitively true too. Similarly, if (14) had been true, then (15) would have been true too (as it happens, (15) is not true). Hence, predicting that $\Diamond\phi$ is true whenever ϕ is true seems correct. So, we fill out the truth table accordingly.

ϕ	1	0
$\Diamond\phi$	1	?

  - But what about when the embedded sentence is false? Let's consider both options:

    FALSE: This leads to the prediction that (13) is false. However that seems incorrect. It seems perfectly (metaphysically) possible that Brian is in Quebec.

    TRUE: This leads to the prediction that (15) is true. However that is clearly incorrect. It is impossible for the number 4 to be prime (given the definition of a prime number).

- In conclusion, simply treating ◇ as a truth functional connective is not going to work in general.

- What about '□':

ϕ	1	0
$\Box\phi$	?	?

- Suppose that (13) is true. Does this mean that it's necessarily true? Clearly not. So, that 'ϕ' is true shouldn't automatically entail that '$\Box\phi$' is true too. So,

ϕ	1	0
$\Box\phi$	0?	?

- But this raises an obvious problem, namely that one cannot conclude that '$\Box\phi$' is false simply from the observation that 'ϕ' is true. For example, the sentence 'bachelors are unmarried' is true, but it is also necessarily true. In conclusion, there is no adequate way of filling out the truth table above.
- In short, we need something more complex than a truth functional analysis to correctly capture the meaning of expressions such as 'it is necessary that' and 'it is possible that'.

- **Vocabulary for Modal Propositional Logic** (MPL):
  The vocabulary of MPL is identical to PL except that we add the two *modal* operators, namely '$\Box$' and '$\Diamond$' to the inventory of expressions. However, we define '$\Diamond$' is defined in terms of '$\Box$'.
  - Sentence Letters: p, q, r, ...
  - Connectives: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$.
  - Parentheses: (,).
  - Modal Operators: $\Box$.

Here, we just pretend that we have defined '$\wedge$', '$\rightarrow$', and '$\leftrightarrow$' in terms of '$\neg$' and '$\vee$'.

- **Additional Modal Operator**: We define $\Diamond$ as follows:
  - $\Diamond\phi =_{def} \neg\Box\neg\phi$

Girle also introduces other operators such as '∇' as a monadic operator for *contingency* where $\nabla\phi =_{df} (\Diamond\phi \wedge \Diamond\neg\phi)$, '$\prec$' as a dyadic operator for strict implication where $(\phi \prec \psi) =_{def} \Box(\phi \rightarrow \psi)$, and '$\circ$' as a dyadic operator for *compatibility* where $(\phi \circ \psi) =_{def} \Diamond(\phi \wedge \psi)$. I will mostly ignore these.

- **Syntax of MPL**
  - Every sentence letter is a well-formed formula (wff).
  - If ϕ and ψ are wffs, then $\neg\phi$, $(\phi \vee \psi)$, $(\phi \wedge \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are also wffs.
  - If ϕ is a wff, then $\Box\phi$ and $\Diamond\phi$ are also wffs.
  - Nothing else is a wff.
- **Semantics**
  - The key difference between PL and MPL is that in MPL, truth values are relativized to possible worlds. Whereas in PL, when giving an interpretation, the interpretation function $\mathcal{I}$ is a function from sentence letters to truth values, in MPL we need a slight more complex function, namely a function from pairs of sentence letters and possible worlds to truth values.
  - And so, for the semantics for MPL, we need something beyond the simple interpretations we used in PL.

  - KRIPKE MODELS
    As is standard, we use a *Kripke model*. A Kripke model $\mathfrak{M}$ is a tuple $\langle \mathbf{F}, \mathcal{I} \rangle$ consisting of a frame $\mathbf{F} = \langle \mathcal{W}, \mathcal{R} \rangle$ and an interpretation function $\mathcal{I}$:

Named after its founder, Saul Kripke.

- $\mathcal{W}$ is the set of all possible worlds, viz. $\mathcal{W} = \{w^1, w^2, w^3, ...\}$
- $\mathcal{R}$ is a binary (accessibility) relation defined on $\mathcal{W}$, viz. $\mathcal{R} \subseteq \mathcal{W}\times\mathcal{W}$
- $\mathcal{I}$ is a 2-place function that assigns 0 or 1 to pairs of sentence letters (p, q, r ...) and worlds (w^1, w^2 ... w^n). I.e. let S be the set of all sentence letters p, q, r, ... then: $\mathcal{I}$: $(S\times\mathcal{W}) \longmapsto \{0,1\}$

That is, $\mathcal{R}$ is a set of pairs of worlds, e.g. $\{\langle w^1,w^2\rangle, \langle w^2,w^3\rangle, \langle w^3,w^1\rangle\}$.

- The frame **F** provides the *structure* of the model, viz. the space of possible worlds $\mathcal{W}$ and the accessibility relations $\mathcal{R} \subseteq \mathcal{W}\times\mathcal{W}$ among the worlds.
- The interpretation function $\mathcal{I}$ assigns semantic values to all the sentence letters relative to worlds.

We are going to talk a lot more about accessibility relations later.

- Next, as we did in PL, we need to define a valuation function $\mathcal{V}$ which permits us to determine the truth values of both simple and complex sentences.

Valuation Function
Where a model $\mathfrak{M} = \langle \mathbf{F}, \mathcal{I}\rangle$, a valuation function $\mathcal{V}$ for $\mathfrak{M}$, $\mathcal{V}_{\mathfrak{M}}$, is defined as the 2-place function that assigns 0 or 1 to each wff relative to each $w \in \mathcal{W}$, subject to the following constraints:

- Where α is any sentence letter and ϕ and ψ are wffs and w is any member of $\mathcal{W}$:

$\mathcal{V}_{\mathfrak{M}}(\alpha,w) = 1$ iff $\mathcal{I}(\alpha,w) = 1$
$\mathcal{V}_{\mathfrak{M}}(\neg\phi, w) = 1$ iff $\mathcal{V}_{\mathfrak{M}}(\phi, w) = 0$
$\mathcal{V}_{\mathfrak{M}}(\phi \vee \psi, w) = 1$ iff $\mathcal{V}_{\mathfrak{M}}(\phi,w) = 1$ or $\mathcal{V}_{\mathfrak{M}}(\psi,w) = 1$
$\mathcal{V}_{\mathfrak{M}}(\Box\phi, w) = 1$ iff For all $w' \in \mathcal{W}$, if $\mathcal{R}(w, w')$, then $\mathcal{V}_{\mathfrak{M}}(\phi,w') = 1$

- Given the definitions of the connectives and '$\Box$', we get the following results:

$\mathcal{V}_{\mathfrak{M}}(\phi \wedge \psi, w) = 1$ iff $\mathcal{V}_{\mathfrak{M}}(\phi,w) = 1$ and $\mathcal{V}_{\mathfrak{M}}(\psi,w) = 1$
$\mathcal{V}_{\mathfrak{M}}(\phi \rightarrow \psi, w) = 1$ iff $\mathcal{V}_{\mathfrak{M}}(\phi,w) = 0$ or $\mathcal{V}_{\mathfrak{M}}(\psi,w) = 1$
$\mathcal{V}_{\mathfrak{M}}(\phi \leftrightarrow \psi, w) = 1$ iff $\mathcal{V}_{\mathfrak{M}}(\phi,w) = \mathcal{V}_{\mathfrak{M}}(\psi, w)$
$\mathcal{V}_{\mathfrak{M}}(\Diamond\phi, w) = 1$ iff There is a $w' \in \mathcal{W}$ such that $\mathcal{R}(w,w') \wedge \mathcal{V}_{\mathfrak{M}}(\phi, w') = 1$

- **Validity and Logical Consequence**
The next step is to define validity and logical consequence for MPL. However, since formulas are not true or false simplicter, the definition of validity needs to be slightly more complicated than it was for PL.

DEFINITION OF VALIDITY IN AN MPL-MODEL:
An MPL-wff ϕ is valid in MPL-model $\mathfrak{M}$ where $\mathfrak{M} = \langle \mathbf{F}, \mathcal{I}\rangle$ and where $\mathbf{F} = \langle \mathcal{W}, \mathcal{R}\rangle$ iff for every $w \in \mathcal{W}$, $\mathcal{V}_{\mathfrak{M}}(\phi,w) = 1$.

- So, that's validity in a model $\mathfrak{M}$. What about validity more generally? Well, that's going to depend on the modal system in question. There are several modal systems each of which give different results with respect to which wffs are valid.
- We are going to start by looking at the "normal" modal systems, namely **K**, **T**, **B**, **S4**, and **S5**.
- For these, we can define validity and logical consequence as follows:

DEFINITION OF MPL-VALIDITY
An MPL-wff is valid in Δ (where Δ is either K, T, B, S4, and S5) iff it is valid in every Δ-model.

DEFINITION OF LOGICAL CONSEQUENCE IN MPL
MPL-wff ϕ is a logical consequence in system Δ of a set of mpl-wffs Γ iff for every Δ-model $\langle \mathcal{W}, \mathcal{R}, \mathcal{I} \rangle$ and each $w \in \mathcal{W}$, if $\mathcal{V}_{\mathfrak{M}}(\gamma, w) = 1$ for each $\gamma \in \Gamma$, then $\mathcal{V}_{\mathfrak{M}}(\phi, w) = 1$

- Next, we will look at semantic trees and countermodels for the simplest of the five systems, namely S5.

Logic 2: Modal Logics – Week 2

Semantic Trees for S5

- Remember that '$\Diamond$' was defined as the dual of '$\Box$', i.e.

$$\Diamond\phi =_{def} \neg\Box\neg\phi$$

- This is analogous to the way in which the existential quantifier is the dual of the universal. And so, for similar reasons, the following equivalences hold:

I.e. $\exists x\phi =_{def} \neg\forall x\neg\phi$

  - $\neg\Diamond\phi \leftrightarrow \Box\neg\phi$
  - $\neg\Box\phi \leftrightarrow \Diamond\neg\phi$

- The aim for today is to establish the rules for constructing semantic trees for the simplest of the normal modal systems S5. In general, I will follow Girle (2009) in using the following notation:
  - '$\phi(w) = 1$' means ϕ is true in world w.
  - '$\phi(w) = 0$' means ϕ is false in world w.

- These conditions determine the following tree construction rules which we refer to as (**MN**)-rules (Modal Negation rules).

1.	$\neg\Diamond\phi$	(w)	$\checkmark$	
$\vdots$	$\vdots$			
n	$\Box\neg\phi$	(w)		l.1, (**MN**)

1.	$\neg\Box\phi$	(w)	$\checkmark$	
$\vdots$	$\vdots$			
n	$\Diamond\neg\phi$	(w)		l.1, (**MN**)

- Next, we introduce rules specifically for standard modal formulas. It's important to note that these rules are specific to the modal system S5. For this reason, we refer to these rules as the ($\Diamond$**S5**)-rule and the ($\Box$**S5**)-rule:

$\Diamond$**S5**-rule

1.	$\Diamond\phi$	(w)	$\checkmark$	
$\vdots$	$\vdots$			
n	ϕ	(v)		l.1, ($\Diamond$**S5**)
		↑ *new world to path*		

NB! For the ($\Diamond$**S5**) rule, it is crucial that the world v introduced in line n is *new* to that part of the tree.

$\Box$**S5**-rule

1.	$\Box\phi$	(w)	$\checkmark$	
$\vdots$	$\vdots$			
n	ϕ	(v)		l.1, ($\Box$**S5**)
		↑ *any world*		

NB! For the ($\Box$**S5**) rule, the world v introduced in line n can be any world including w.

- The construction rules introduced in lecture 1, remain basically the same although they need to be to be revised to account for the fact that formulas are true/false relative to worlds. So, we revise as follows:

Propositional Logic Tree Rules (PTR) for MPL

$$\neg\neg\phi \quad (w)$$
$$|$$
$$\phi \quad (w)$$

$$\phi \wedge \psi \quad (w)$$
$$|$$
$$\phi \quad (w)$$
$$\psi \quad (w)$$

$$\neg(\phi \wedge \psi) \quad (w)$$
$$\neg\phi\ (w) \qquad \neg\psi\ (w)$$

$$(\phi \vee \psi) \quad (w)$$
$$\phi\ (w) \qquad \psi\ (w)$$

$$\neg(\phi \vee \psi) \quad (w)$$
$$|$$
$$\neg\phi \quad (w)$$
$$\neg\psi \quad (w)$$

$$(\phi \rightarrow \psi) \quad (w)$$
$$\neg\phi\ (w) \qquad \psi\ (w)$$

$$\neg(\phi \rightarrow \psi) \quad (w)$$
$$|$$
$$\phi\ (w)$$
$$\neg\psi\ (w)$$

$$(\phi \leftrightarrow \psi) \quad (w)$$
$$\phi\ (w) \qquad \neg\phi\ (w)$$
$$\psi\ (w) \qquad \neg\psi\ (w)$$

$$\neg(\phi \leftrightarrow \psi) \quad (w)$$
$$\phi\ (w) \qquad \neg\phi\ (w)$$
$$\neg\psi\ (w) \qquad \psi\ (w)$$

- Note that the MN and PTR rules are all "single-world" rules. That is, applying any of these rules never lead to a change of evaluation world.
- Let's now put these construction rules to work. We will start by showing that the set of premises $\{\Box(p \rightarrow q), \neg\Diamond q\}$ entails the conclusion $\neg p$:

(A1) $\{\Box(p \rightarrow q), \neg\Diamond q\} \vDash_{S5} \neg p$

- **Semantic Tree for (A1)**

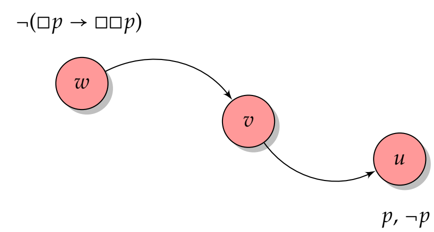
1.	$\Box(p \to q)$	(w)	√	Premise
2.	$\neg\Diamond q$	(w)	√	Premise
3.	$\neg\neg p$	(w)	√	Negated Conclusion
4.	p	(w)		l.3
5.	$\Box\neg q$	(w)	√	l.2, **MN**
6.	$\neg q$	(w)		5, $\Box$**S5**
7.	$(p \to q)$	(w)	√	1, $\Box$**S5**
8.	$\neg p$ (w) × q (w) ×			l.7

- Let's look at another one:

(1) $\vDash_{S5} \Box p \to \Box\Box p$

- Proof.

1.	$\neg(\Box p \to \Box\Box p)$	(w)	√	NTF
2.	$\Box p$	(w)	√	l.1
3.	$\neg\Box\Box p$	(w)	√	l.1
4.	$\Diamond\neg\Box p$	(w)	√	l.3, **MN**
5.	$\neg\Box p$	(v)	√	l.4, $\Diamond$**S5**
6.	$\Diamond\neg p$	(v)	√	l.5, **MN**
7.	$\neg p$	(u)		l.7, $\Diamond$**S5**
8.	p	(u)		l.2, $\Box$**S5**
	×			



- **Counterexamples and Countermodels in MPL**

In MPL, formulas are true relative to worlds, so an atomic sentence might have one truth value relative to one world and a different truth value relative to another.

	w	v	u	$\cdots$
p	1	0	0	
q	0	1	0	
r	1	1	1	
$\vdots$				

- Moreover, remember that for the normal modal systems in MPL, validity is defined as follows: a formula ϕ is valid iff it is true in every world of every model of the system. So, in order to establish a counterexample to a formula, one needs only demonstrate that the formula is false in at least one world of a relevant model. For example, if the system in question in S5, then a counterexample requires an S5 model.

When we start considering accessibility relations, it will be easier to appreciate the difference between an S5-model and models for other normal systems.

- Let's consider an example.

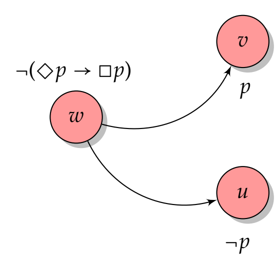
(2) $\nvDash_{S5} \Diamond p \rightarrow \Box p$

- A countermodel to this formula requires a model $\mathfrak{M}$ where the antecedent ('$\Diamond p$') is true and that the consequent ('$\Box p$') is false.
- A countermodel contains the following ingredients:
  (a) A specification of a set of possible worlds $\mathcal{W}$.
  (b) A specification of the relevant accessibility relation(s).
  (c) A specification of an interpretation function taking pairs of atomic sentences and worlds as arguments and truth values as outputs.
- To simplify, we ignore ignore (b) for the present example. Here is a countermodel to (2):

S5-model $\mathfrak{M}$: $\mathcal{W}$: $\{w, v\}$
$\mathcal{I}$: $\{\langle\langle p,w\rangle, 0\rangle, \langle\langle p, v\rangle, 1\rangle\}$

- In this model, 'p' is true at v, but false at w. This suffices to generate a counterexample to (2). If 'p' is true at v, then '$\Diamond p$' is true at w. However, since 'p' is false at w, '$\Box p$' is false at w.
- As is the case in PL, one can search for a countermodel using a semantic tree.

1.	$\neg(\Diamond p \rightarrow \Box p)$	(w)	√	NTF
2.	$\Diamond p$	(w)	√	l.1
3.	$\neg\Box p$	(w)	√	l.1
4.	$\Diamond\neg p$	(w)	√	l.3, **MN**
5.	p	(v)		l.2, $\Diamond$**S5**
6.	$\neg p$	(u)		l.4, $\Diamond$**S5**
	↑			



- Note, there is no contradiction between l.5 and l.6 as these formulas are not evaluated at the same worlds.

World-Generators and World-Fillers

- Above we assumed the rules $\Diamond$**S5** and $\Box$**S5**. These can be characterized as *world-generator* and *world-filler* rules respectively.
  - The rule $\Diamond$**S5** is a world-generator rule since whenever we encounter a true possibility statement, we unpack the embedded formula in a *new* world to the path.
  - By contrast, $\Box$**S5** is a world-generator rule since whenever we encounter a true necessity statement, we unpack this by adding the embedded formula to one (or more) already existing worlds to the path.
- The $\Box$**S5** rule is however completely unrestricted: It allows us to unpack the embedded formula in *any* world whatsoever.
- So, one way of changing the logic is to restrict the world-filler rule. We will now consider a variety of ways of doing this.

Accessibility

- In general, we will say that a world w has access to another world v only if v has somehow been generated from w. To indicate that a world w has access to another world v, we write wRv.

Accessibility relations are sometimes annotated as follows: 'Rwv' or '$R(w, v)$'.

- Moreover, we refine the rules for '$\Diamond$' and '$\Box$' in the following way:
  - If $\Diamond\phi(w) = 1$, then there is a world v accessible from w such that '$\phi(v)$' = 1.
  - If $\Box\phi(w) = 1$, then for all worlds v accessible from w '$\phi(v)$' = 1.
- And this yields the following general truth conditions:
  - $\Diamond\phi(w) = 1$ iff $\exists v(wRv \wedge \phi(v) = 1)$
  - $\Box\phi(w) = 1$ iff $\forall v(wRv \rightarrow \phi(v) = 1)$
- Now, to keep track of the relevant accessibility relations in a tree, we need to modify the modal tree rules:

$\Diamond$**R**-rule

1.	$\Diamond\phi$	(w)	✓	
⋮	⋮			
n	wRv			
n+1	ϕ	(v)		l.1, ($\Diamond$**R**)
		↑ *new world to path*		

NB! For the ($\Diamond$**R**) rule, it is still crucial that the world v introduced in line n is *new* to that part of the tree.

$\Box$**R**-rule

1.	$\Box\phi$	(w)	✓	
2.	wRv			
⋮	⋮			
n	ϕ	(v)		l.1-2, ($\Box$**R**)

NB! For the ($\Box$**R**) rule, notice that antecedent access to some world is *required* in order to "discharge" the box.

Other Modal Systems

- So far we have introduced the following construction rules:
  - **PTR**: Propositional Logic Tree Rules for MPL (cf. p.2)
  - **MN**: Modal Negation Rules (cf. p.1)
  - $\Box$**S5**, $\Diamond$**S5** (cf. p.1)
  - $\Box$**R**, $\Diamond$**R** (cf. p.5)
- The tree construction rules for every modal system is going to include **PTR** and **MN** (as these are single-world rules). So, the tree rules for the system **S5** (**S5Tr**) is going to be the union of **PTR**, **MN**, and the **S5**-rules: viz. (**S5Tr**) = **PTr** ∪ **MN** ∪ {$\Diamond$**S5**, $\Box$**S5**}
- We will now consider the standard range of systems weaker than **S5**, namely **K**, **T**, **S4**, and **B**.

System K

- The tree rules for **K** (**KTr**) is simply **PTR** $\cup$ **MN** $\cup$ {◇**R**, □**R**}.
- It is noticeable just how much weaker this system is than **S5**. For example in **K** some of even the most simple modal statements that it might seem *should* be valid are invalid. Consider, for example, (3) below.

(3) $\Box p \rightarrow p$

- To construct a countermodel for this wff in **K**, let's start by constructing a tree:

1.	$\neg(\Box p \rightarrow p)$	(w)	✓	NTF
2.	$\Box p$	(w)		l.1
3.	$\neg p$	(w)		l.1
	↑			

- When we only have the □**R**-rules are our disposal, we can only discharge a box if there is an antecedently accessible world. And since there is no such world available, we get stuck almost immediately.
- So, a simple countermodel will look as follows:
  K-model $\mathfrak{M}$: $\mathcal{W}$: $\{w\}$
  $\mathcal{R}$: $\varnothing$
  $\mathcal{I}$: $\{\langle\langle p,w\rangle, 0\rangle\}$
- In other words, this means that in the system **K**, one cannot even prove that if it is necessary that ϕ, then ϕ is true.

Notice that $\Box\phi(w)$ is true when there are no accessible worlds from w. This follows from the semantics for $\Box\phi(w)$:

$\Box\phi(w) = 1$ iff $\forall v(wRv \rightarrow \phi(w) = 1)$

Since the antecedent of this conditional is false whenever w has access to no worlds, the conditional is true, and so $\Box\phi(w)$ is true.

System T

- One way to increase the power of the system (which will ensure the validity of '$\Box\phi \rightarrow \phi$') is to stipulate specific properties of the accessibility relation. For example, we might stipulate that the accessibility relation is *reflexive*:

Reflexivity is formally speaking a property of relations. Specifically, if a relation R is reflexive, then the following holds: $\forall x(Rxx)$

(Refl.)	⋮	⋮	
	n	wRw	**Refl.**
		↑	
		for any w in the path	

- If we add **Refl.** to **KTr** we get the modal system **T**. The tree rules for **T** are thus the following: (**TTr**) = **PTR** $\cup$ **MN** $\cup$ {□**R**, ◇**R**, **Refl**}
- It's easy to see that with **TTr**, the formula in (3) is valid:

1.	$\neg(\Box p \rightarrow p)$	(w)	√	NTF
2.	$\Box p$	(w)	√	l.1
3.	$\neg p$	(w)		l.1
4.	wRw			**Refl.**
5.	p	(w)		$\Box$**R**
	×			

- Hence, (3) is valid in **T**, but invalid in **K**.

System S4

- To get the system S4, we impose the further requirement that the accessibility be *transitive*:

Transitivity is formally speaking a property of relations. Specifically, if a relation R is transitive, then the following holds: $\forall x \forall y \forall z\big((Rxy \wedge Ryz) \rightarrow Rxz\big)$

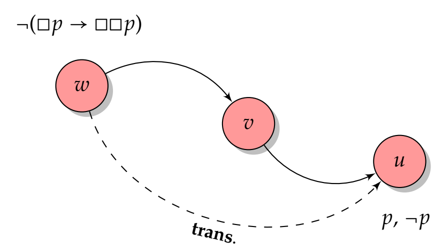
(Trans.)	1.	wRv	
	2.	vRu	
	⋮	⋮	
	n	wRu	l.1, l.2, **Trans.**

- The tree rules for **S4** are thus as follows:
  (S4Tr) = **PTR** ∪ **MN** ∪ {$\Diamond$**R**, $\Box$**R**, **Refl**, **Trans**}
- Transitivity is needed in order for the formula in (4) to come out valid:

(4) $\Box p \rightarrow \Box\Box p$

- Let's look at the proof.

1.	$\neg(\Box p \rightarrow \Box\Box p)$	(w)	√	NTF
2.	$\Box p$	(w)	√	l.1
3.	$\neg\Box\Box p$	(w)	√	l.1
4.	$\Diamond\neg\Box p$	(w)	√	l.3, **MN**
5.	wRv			l.4, $\Diamond$**R**
6.	$\neg\Box p$	(v)	√	l.4, $\Diamond$**R**
7.	$\Diamond\neg p$	(v)	√	l.6, **MN**
8.	vRu			l.7, $\Diamond$**R**
9.	$\neg p$	(u)		l.7, $\Diamond$**R**
10.	wRu			l.5, l.8, **Trans**
11.	p	(u)		l.2, l.10, $\Box$**R**
	×			



- Notice that **Trans** is crucial here. Without this rule, it would not be possible to discharge the box in line 2 in u which is needed to generate a contradiction.
- So, (3) is valid in **S4**, but invalid in **K** and **T**.

System B

- If assume that the accessibility relation is *symmetric* (in addition to being *reflexive*), we end up with the system **B**.

This system is also sometimes referred to as **Br**. It's named after its inventor, logician E.J. Brouwer.

- When *symmetry* is assumed, we get the following tree rule:

Symmetry is formally speaking a property of relations. Specifically, if a relation R is symmetric, then the following holds: $\forall x \forall y(Rxy \rightarrow Ryx)$

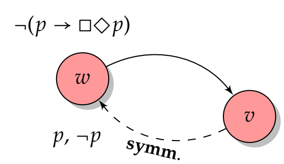
(Symm.)	1.	wRv	
	$\vdots$	$\vdots$	
	n	vRw	l.1, Symm.

- The formula which characterizes the system **B** is the wff in (5).

(5) $p \rightarrow \Box\Diamond p$

- That (5) is valid in **B** can be proved as follows:

1.	$\neg(p \rightarrow \Box\Diamond p)$	(w)	√	NTF
2.	p	(w)		l.1
3.	$\neg\Box\Diamond p$	(w)	√	l.1
4.	$\Diamond\neg\Diamond p$	(w)	√	l.3, **MN**
5.	wRv			l.4, $\Diamond$**R**
6.	$\neg\Diamond p$	(v)	√	l.4, $\Diamond$**R**
7.	$\Box\neg p$	(v)	√	l.6, **MN**
8.	vRw			l.5, **Symm.**
9.	$\neg p$	(w)		l.7, l.8, $\Box$**R**
	$\times$			



- The tree rules for **B** are thus the following:
  (**BTr**) = **PTR** ∪ **MN** ∪ {$\Diamond$**R**, $\Box$**R**, **Refl.**, **Symm.**}

Back to System S5

- If the accessibility relation is assumed to be reflexive, transitive, *and* symmetric, we then end up with the system with which we started **S5**.

A relation that is reflexive, transitive, and symmetric is also known as an *equivalence* relation.

- So, instead of characterizing **S5** in terms of special dedicated rules for $\Diamond$ and $\Box$, we can think of **S5** simply as a system that includes the following rules:
  (**S5Tr**) = **PTR** ∪ **MN** ∪ {$\Diamond$**R**, $\Box$**R**, **Refl.**, **Trans.**, **Symm.**}
- Hence, in **S5**, the formulas in (3), (4), and (5) are all valid:
  - $\vDash_{S5} \Box p \rightarrow p$ (Refl.)
  - $\vDash_{S5} \Box p \rightarrow \Box\Box p$ (Trans.)
  - $\vDash_{S5} p \rightarrow \Box\Diamond p$ (Symm.)
- Moreover, in **S5** the following is also valid (which is not valid in any other system):
  - $\vDash_{S5} \Diamond p \rightarrow \Box\Diamond p$

EXERCISE: Prove this.

Frames

- This brings us back to the notion of a *frame* **F** which was introduced when we talked about Kripke models. Remember, a frame **F** is a pair of a set of worlds $\mathcal{W}$ and an accessibility relation $\mathcal{R}$.
  - So, if the accessibility relation $\mathcal{R}$ in **F** is reflexive, we say that it is a **T**-frame.
  - And if the accessibility relation $\mathcal{R}$ in **F** is reflexive and transitive, we say that it is an **S4**-frame.
  - And If the accessibility relation $\mathcal{R}$ in **F** is reflexive and symmetric, we say that it is a **B**-frame.
  - And, finally, if the accessibility relation $\mathcal{R}$ in **F** is reflexive, transitive, and symmetric, we say that it is an **S5**-frame.

Strictly speaking $\mathcal{R}$ is a subset of the cartesian product of $\mathcal{W}$: $\mathcal{R} \subseteq (\mathcal{W} \times \mathcal{W})$.

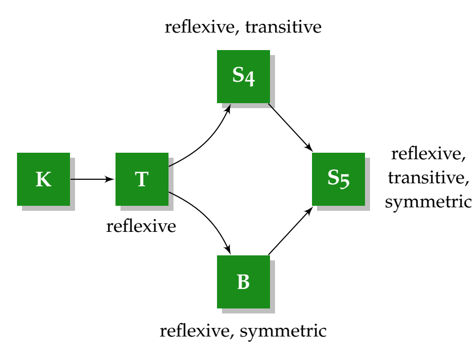
The Relation between the Normal Modal Systems

- The tree rules for each of the normal modal systems thus look as follows:

KTr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**}
TTr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**, **Refl.**}
S4Tr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**, **Refl.**, **Trans.**}
BTr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**, **Refl.**, **Symm.**}
S5Tr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**, **Refl.**, **Trans.**, **Symm.**}

- It follows that from these relations between the normal modal systems that:
  - All **K**-valid formulas are **T**-valid.
  - All **T**-valid formulas are **S4**-valid.
  - All **T**-valid formulas are **B**-valid.
  - All **S4**-valid formulas are **S5**-valid.
  - All **B**-valid formulas are **S5**-valid.

These inferential relations between the systems can be diagramatically represented as follows:



Logic 2: Modal Logics – Week 4

Natural Deduction in PL

- A proof in natural deduction consists of a sequence of sentences which are either premises, assumptions, or derived from previous sentences via various inference rules or replacement rules. The last line of a sequence serves as the conclusion of the proof.

- Every line in a natural deduction proof must be numbered on the left and given a justification on the right. For example, a proof of (1) would look as follows:

(1) $p, (p \to q), (q \to r) \vdash_{PL} r$

1	p	premise
2	$p \to q$	premise
3	$q \to r$	premise
4	q	MP, 1, 2
5	r	MP, 3, 4

- This proof of r relies on the premises in lines 1, 2, and 3, and on applications of an inference, namely modus ponens. So, this is a proof that from the premises in 1 to 3 and the rule of modus ponens, one can derive the conclusion r.

Various inference rules, including modus ponens, are explicated below

- Another example:

(2) $(p \to q), (q \to r), \neg r \vdash_{PL} \neg p$

1	$p \to q$	premise
2	$q \to r$	premise
3	$\neg r$	premise
4	$\neg q$	MT, 2, 3
5	$\neg p$	MT, 1, 2

- This proof relies on the premises in line 1 and 2, and on applications of the inference rule of modus tollens (cf. below).

- Different natural deduction systems will assume different basic inference and replacement rules. We are going to be quite generous in this regard and assume a fairly extensive list of these.

- **Replacement Rules**: A replacement is a rule that allows the replacement of one formula for another. For example, the two so-called DeMorgan replacement rules look as follows:

RULE	NAME	ABBREVIATION
$(\neg\phi \vee \neg\psi) :: \neg(\phi \wedge \psi)$	DeMorgan	(DeM)
$(\neg\phi \wedge \neg\psi) :: \neg(\phi \vee \psi)$	DeMorgan	(DeM)

- These rules simply say that the formulas flanking the '::' may be substituted for one another at any point in a proof. When using a replacement rule, one must however, as usual, cite the line on which the replacement relies and the name of the rule. Here is an example:

(3) $(\neg p \wedge \neg q), (\neg(p \vee q) \rightarrow \neg q) \vdash_{\text{PL}} \neg q$

1	$\neg p \wedge \neg q$	premise
2	$\neg(p \vee q) \rightarrow \neg q$	premise
3	$\neg(p \vee q)$	DeM, 1
4	q	MP, 2, 3

- Here is the list of replacement rules we will assume:

If we were being more rigorous, we would *prove* for each of these replacement rules that they follow from more basic rules of inference.

For example, we can prove the commutativity of conjunction using only the inference rules **Conj** and **Simp**, cf. below.

EXERCISE (easy): Prove the commutativity of conjunction.

EXERCISE (harder): Prove the commutativity of disjunction.

- **Replacement Rules**

$\neg\neg\phi :: \phi$	double negation	(DN)
$(\phi \wedge \psi) :: (\psi \wedge \phi)$	commutativity for $\wedge$	(Com)
$(\phi \vee \psi) :: (\psi \vee \phi)$	commutativity for $\vee$	(Com)
$(\phi \wedge (\psi \wedge \chi)) :: (\phi \wedge \psi) \wedge \chi)$	associativity for $\wedge$	(Assoc)
$(\phi \vee (\psi \vee \chi)) :: (\phi \vee \psi) \vee \chi)$	associativity for $\vee$	(Assoc)
$(\phi \wedge (\psi \vee \chi)) :: ((\phi \wedge \psi) \vee (\phi \wedge \chi))$	distribution	(Dist)
$(\phi \vee (\psi \wedge \chi)) :: ((\phi \vee \psi) \wedge (\phi \vee \chi))$	distribution	(Dist)
$(\neg\phi \vee \neg\psi) :: \neg(\phi \wedge \psi)$	DeMorgan	(DeM)
$(\neg\phi \wedge \neg\psi) :: \neg(\phi \vee \psi)$	DeMorgan	(DeM)
$(\phi \vee \phi) :: \phi$	idempotence	(Idem)
$(\phi \wedge \phi) :: \phi$	idempotence	(Idem)
$(\phi \rightarrow \psi) :: (\neg\phi \vee \psi)$	material implication	(IMP)
$(\phi \rightarrow \psi) :: (\neg\psi \rightarrow \neg\phi)$	contraposition	(Cont)
$((\phi \wedge \psi) \rightarrow \chi) :: (\phi \rightarrow (\psi \rightarrow \chi))$	exportation	(Exp)
$(\phi \rightarrow (\psi \rightarrow \chi)) :: (\psi \rightarrow (\phi \rightarrow \chi))$	permutation	(Per)
$(\phi \leftrightarrow \psi) :: ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$	equivalence	(Equiv)
$(\phi \leftrightarrow \psi) :: ((\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi))$	equivalence	(Equiv)
$\phi :: (\phi \wedge (\psi \vee \neg\psi))$	taut. conj.	(TConj)
$\phi :: (\phi \vee (\psi \wedge \neg\psi))$	taut. disj.	(TDisj)

- **Inference Rules**: An inference rule is a rule permitting the inference of a formula (or formulas) on the basis of another formula (or formulas). For example, the rule of *modus ponens* says that if ϕ and $(\phi \rightarrow \psi)$ are lines in a proof, one may then add ψ as a line too.
- We will assume the following inference rules:

$$\frac{\begin{array}{c}(\phi \rightarrow \psi)\\ \phi\end{array}}{\psi}$$

modus ponens (**MP**)

$$\frac{\begin{array}{c}(\phi \rightarrow \psi)\\ \neg\psi\end{array}}{\neg\phi}$$

modus tollens (**MT**)

$$\frac{(\phi \wedge \psi)}{\begin{array}{c}\phi\\ \psi\end{array}}$$

simplication (**Simp**)

$$\frac{\begin{array}{c}\phi\\ \psi\end{array}}{(\phi \wedge \psi)}$$

conjunction (**Conj**)

$$\frac{\phi}{(\phi \vee \psi)} \qquad \frac{\phi}{(\psi \vee \phi)}$$

addition (**Add**)

$$\frac{\begin{array}{c}(\phi \vee \psi)\\ \neg\phi\end{array}}{\psi} \qquad \frac{\begin{array}{c}(\phi \vee \psi)\\ \neg\psi\end{array}}{\phi}$$

disjunctive syllogism (**DS**)

$$\frac{\begin{array}{c}(\phi \rightarrow \psi)\\ (\psi \rightarrow \chi)\end{array}}{(\phi \rightarrow \chi)}$$

hypothetical syllogism (**HS**)

$$\frac{\begin{array}{c}(\phi \rightarrow \psi)\\ (\chi \rightarrow \delta)\\ (\phi \vee \chi)\end{array}}{(\psi \vee \delta)}$$

constructive dilemma (**CD**)

1	$\quad\phi$	assumption
2	$\quad\vdots$	
3	$\quad\psi$	
4	$\phi \rightarrow \psi$	CP

conditional proof

1	$\quad\phi$	assumption
2	$\quad\vdots$	
3	$\quad\bot$	
4	$\neg\phi$	RAA

reductio ad absurdum

- **Conditional Proofs**
  One of the inference rules above is the rule a *conditional proof*. A conditional proof is just that, i.e. a proof of a conditional statement.

- Conditional proofs ordinarily make use of *assumptions* which are importantly different from *premises*. An assumption is, in a sense, a temporary premise. That is, it is a line in a proof that serves to derive certain consequences, but which (in contrast to a premise) must be "discharged" (i.e. discarded) before the end of the proof. This is to ensure that the conclusion of the proof does not rely on mere assumptions.

- One way to discharge an assumption is by using *conditional proof* (CP). This rule says that if from an assumption ϕ, you can derive ψ (say, by using various replacement and inference rules), you have then given a conditional proof (a proof with a conditional conclusion), namely that $(\phi \rightarrow \psi)$ follows from *no* assumptions.

- Since discharging assumptions is crucial, keeping track of these is imperative. To do this, we use the following notation: When making an assumption, we add an indented vertical line (to indicate that we are starting a *subproof* that makes use of an assumption). Moreover, we underline the relevant assumption to indicate that this what we are currently aiming to discharge before returning to the main proof.

- Let's consider an example.

This is effectively a proof of the inference rule *hypothetical syllogism*, so we will assume that we are not allowed to use this rule for this proof.

(4) $(p \rightarrow q), (q \rightarrow r) \vdash_{\text{PL}} (p \rightarrow r)$

1	$p \rightarrow q$	premise
2	$q \rightarrow r$	premise
3	$\quad \mid \underline{p}$	assumption
4	$\quad \mid q$	MP, 1, 3
5	$\quad \mid r$	MP, 2, 4
6	$p \rightarrow r$	CP, 3, 5

- Proofs using assumptions can also be used to prove formulas from no premises (or the empty set of premises). For example:

(5) $\vdash_{\text{PL}} p \rightarrow (p \vee q)$

1	$\quad \mid \underline{p}$	assumption
2	$\quad \mid p \vee q$	Add, 1
3	$p \rightarrow (p \vee q)$	CP, 1, 2

- This proof shows that from the assumption p, one can derive $(p \lor q)$ using the inference rule *addition*. Consequently, we are licensed to conclude that $(p \to (p \lor q))$ is a tautology, i.e. it follows from the empty set of premises (viz. no premises and no assumptions).

- **Indirect Proofs**
  Another method of proof is *indirect proofs*. An indirect proof proceeds by assuming the negation of the target formula and then demonstrating (using replacement and inference rules) that this leads to a contradiction. From the derivation of a contradiction, one then infers the negation of the initial assumption (which thereby discharges the assumption), and this then shows that the target formula is true.

Indirect proofs are also commonly referred to as *proof by contradiction* or simply *reductios*.

- Here is an example:

(6) $\vdash_{\text{PL}} (p \lor \neg p)$

1	│ $\neg(p \lor \neg p)$	assumption
2	│ $\neg p \land \neg\neg p$	DeM, 1
3	│ $\neg p$	Simp., 2
4	│ $\neg\neg p$	Simp., 2
5	│ $\bot$	3, 4
6	$\neg\neg(p \lor \neg p)$	RAA, 1, 5
7	$(p \lor \neg p)$	DN, 6

- Also, it is important to note that strictly speaking any conclusion will follow from a contradiction. This is easily proved.

(7) $(p \land \neg p) \vdash_{\text{PL}} q$

1	$p \land \neg p$	premise
2	│ $\neg q$	assumption
3	│ p	Simp., 1
4	│ $\neg p$	Simp., 1
5	│ $\bot$	3, 4
6	$\neg\neg q$	RAA, 2, 5
7	q	DN, 6

Natural Deduction in MPL

- Next, we need to extend the natural deduction system for PL to MPL. However, we will only consider natural deduction systems for the logics **T**, **S4**, and **S5**.
- We start by assuming the following modal replacement rules and modal rules of inference.
- **Modal Replacement Rules**

$\neg\Box\phi :: \Diamond\neg\phi$	modal negation	(MN)
$\neg\Diamond\phi :: \Box\neg\phi$	modal negation	(MN)
$\neg\Box\neg\phi :: \Diamond\phi$	modal negation	(MN)
$\neg\Diamond\neg\phi :: \Box\phi$	modal negation	(MN)

- **Modal Rules of Inference**
- In addition to the inference rules for PL above, we assume the following modal inference rules:

$$\frac{\phi}{\Diamond\phi}$$
possibility introduction (**PI**)

$$\frac{\Box\phi}{\phi}$$
modal reiteration T (**MRT**)

$$\frac{\Box\phi}{\Box\phi}$$
modal reiteration (**MRS4**)

$$\frac{\Diamond\phi}{\Diamond\phi}$$
modal reiteration S5 (**MRS5**)

Note: The MR-rules are system specific:

For a derivation in **T**, one may only use **MRT**.

For a derivation in **S4**, one may only use **MRT** and **MRS4**.

For a derivation in **S5**, one may use **MRT**, **MRS4**, and **MRS5**.

- **Null Assumption Proofs**
  In addition to the modal inference rules, one extra ingredient is needed, namely the notion of *null assumption proof*.
- A null assumption proof has the following features:

1. There is *no assumption* (or a null assumption).
2. Every formula inside the scope of a null assumption must be deduced from preceeding formulas *inside the scope of* that null assumption unless it is deduced by modal reiteration for the appropriate modal logic.
3. The proof ends with a discharge of the null assumption line.

4. After the discharge of the null assumption line where the formula immediately before the discharge is ϕ, the next formula is $\Box\phi$.

- This yields the following inference rule for null assumption proofs.

1	$\quad\vert\ \vert\ \underline{\quad}$	null assumption
2	$\quad\vert\ \vert\ \vdots$	
3	$\quad\vert\ \vert\ \phi$	
4	$\quad\vert\ \Box\phi$	NI

necessity introduction

- In other words, in combination with the MR-rules above, a null assumption proof allows us to adjoin boxes to formulas.
- Let's consider a couple of simple examples of derivations that use the modal inference rules.

(8) $\vdash_T \Box\Box(p \to \neg q) \to (\neg p \vee \neg q)$

1	$\vert\ \vert\ \underline{\Box\Box(p \to \neg q)}$	assumption
2	$\vert\ \vert\ \Box(p \to \neg q)$	MRT, 1
3	$\vert\ \vert\ p \to \neg q$	MRT, 2
4	$\vert\ \vert\ (\neg p \vee \neg q)$	IMP, 3
5	$\vert\ \Box\Box(p \to \neg q) \to (\neg p \vee \neg q)$	CP, 1–4

(9) $\vdash_T \Box p \to \Box\neg\neg p$

1	$\vert\ \vert\ \underline{\Box p}$	assumption
2	$\vert\ \vert\ \vert\ \underline{\quad}$	**null assumption**
3	$\vert\ \vert\ \vert\ p$	MRT, 1
4	$\vert\ \vert\ \vert\ \neg\neg p$	DN, 3
5	$\vert\ \vert\ \Box\neg\neg p$	NI, 2–4
6	$\vert\ \Box p \to \Box\neg\neg p$	CP, 1–5

(10) $\vdash_{S5} \Box p \rightarrow \Box\Diamond\Diamond\Box p$

1	$\Box p$	assumption
2	$\Diamond\Box p$	PI, 1
3	$\Diamond\Diamond\Box p$	PI, 2
4		**null assumption**
5	$\Diamond\Diamond\Box p$	MRS5, 3
6	$\Box\Diamond\Diamond\Box p$	NI, 4, 5
7	$\Box p \rightarrow \Box\Diamond\Diamond\Box p$	CP, 1–5

- One thing that might not be entirely obvious is how, given the rules available, we are supposed to prove “bare” necessities such as (11) and (12).

(11) $\Box(p \rightarrow p)$
(12) $\Box(p \rightarrow (p \vee q))$

- Remember to introduce a $\Box$, we need to use a null hypothesis proof, but since the only lines in a null hypothesis proof must be derived from previous lines (within the null assumption proof) or by MRT rules, it is not obvious what to do here.
- The solution that we will adopt here is to assume that it is permitted to have a line in a null assumption proof that is not derived from previous lines or by MRT rules *given that the line has been derived by doing a subproof within the null assumption proof and that the line relies on no assumptions.*
- Here is an example.

(13) $\vdash_{T} \Box(p \rightarrow p)$

1		**null assumption**
2	p	assumption
3	$\neg p$	assumption
4	$\bot$	2, 3
5	$\neg\neg p$	RAA, 2, 3
6	p	DN, 5
7	$p \rightarrow p$	CP, 2–6
8	$\Box(p \rightarrow p)$	NI, 1, 7

Logic 2: Modal Logics – Week 5

Predicate Logic – A Recap

- We start today with a recap of the syntax and semantics of **First Order Logic** (FOL). First we need a vocabulary (or a lexicon) for our formal language $\mathfrak{L}$.

Primitive Vocabulary of $\mathfrak{L}$

- The language $\mathfrak{L}$ contains the following primitive expression types.
  - **Connectives**: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
  - **Individual Variables**: x, y, z, ...
  - **Individual Constants**: a, b, c, ...
  - **Predicate Constants**: F_1, F_2, F_3 ... R_1, R_2, R_3 ...
  - **Quantifiers**: $\forall$, $\exists$
  - **Parentheses**: (,)
- Variables and constants are both referred to as *terms*.

We will add numerical superscripts to variables and constants when we need more than the alphabet provides. We will use F for 1-place predicates, and R for relations. However, it is assumed that our vocabulary also contains predicates of higher arity.

Syntax of $\mathfrak{L}$

- Next, we state the syntactic rules of $\mathfrak{L}$, i.e. the rules that determine whether a string of $\mathfrak{L}$ is a well formed formula (wff).

1. If Π is an n-place predicate and $\alpha_1 ... \alpha_n$ are terms, then $\Pi\alpha_1 ... \alpha_n$ is a well-formed formula (wff).
2. If ϕ and ψ are wffs, and α is a variable, then the following are wffs:

$$\neg\phi \mid (\phi \wedge \psi) \mid (\phi \vee \psi) \mid (\phi \rightarrow \psi) \mid (\phi \leftrightarrow \psi) \mid \forall\alpha\phi \mid \exists\alpha\phi$$

3. Only strings formed on the basis of 1. and 2. are wffs.

Variable Binding

- A formula is *closed* if and only if all of its variables are bound. The notion of binding is defined as follows.

> BINDING: An occurrence of a variable α in a wff ϕ is bound in ϕ iff α is within an occurrence of some wff of the form $\forall\alpha\psi$ or $\exists\alpha\psi$ within ϕ. Otherwise α is free.

- For example, in the formulas below, x is bound but y is free.

$\exists x F y \mid \forall x x R y \mid \forall x(Fx \wedge Fy) \mid \forall x \exists y Fx \wedge Fy$

- Notice that both open and closed formulas count as wffs.
- Finally, notice that $\forall$ and $\exists$ are duals.

$$\forall\alpha\phi \Leftrightarrow \neg\exists\alpha\neg\phi \quad | \quad \exists\alpha\phi \Leftrightarrow \neg\forall\alpha\neg\phi$$

Semantics and Models for $\mathfrak{L}$

- Next, we need a semantics for $\mathfrak{L}$, i.e. a compositional method for determining the conditions for the wffs of $\mathfrak{L}$. This requires a model for $\mathfrak{L}$.
- A **model** $\mathfrak{M}$ is an ordered pair $\langle\mathcal{D}, \mathcal{I}\rangle$, where:
  - $\mathcal{D}$ is a non-empty set (the domain).
  - $\mathcal{I}$ is an interpretation function which satisfies the following two conditions:
    1. if α is a constant, then $\mathcal{I}(\alpha) \in \mathcal{D}$.
    2. if Π is an n-place predicate, then $\mathcal{I}(\Pi)$ is an n-place relation over $\mathcal{D}$.
- In short, the model provides an extension (i.e. meaning) of the non-logical constants, viz. individual constants and predicate constants.
- Next, we require a **recursive definition of truth** for the wffs of $\mathfrak{L}$, but since our vocabulary now includes both variables and quantifiers, we need to say something about the interpretation of these expressions.

Variables in $\mathfrak{L}$

- A variable assignment g for a model $\mathfrak{M}$ is a function from variables in the object language $\mathfrak{L}$ to objects in the domain $\mathcal{D}$. I.e. let $\mathcal{G}$ be the set of variables, then:

$$g\colon \mathcal{G} \longmapsto \mathcal{D}$$

- Here is an example of a variable assignment g:

$$g = \begin{bmatrix} x & \longrightarrow & \text{Pluto} \\ y & \longrightarrow & \text{The Dugald Stewart Building} \\ z & \longrightarrow & \text{Theresa May} \\ \vdots & & \vdots \end{bmatrix}$$

- It is important to distinguish between the interpretation of constants and variables: Constants are interpreted relative to the interpretation function $\mathcal{I}$ in $\mathfrak{M}$ whereas variables are interpreted relative to some variable assignment g for $\mathfrak{M}$.
- Let $\mathcal{V}^{\mathfrak{M},g}(\alpha)$ be the valuation of a term α relative to $\mathfrak{M}$ and g. If so,

$$\mathcal{V}^{\mathfrak{M},g}(\alpha) = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a } \textit{constant} \\ g(\alpha) & \text{if } \alpha \text{ is a } \textit{variable} \end{cases}$$

Valuations and Truth-in-a-Model

- More generally, a valuation function $\mathcal{V}$ for a model $\mathfrak{M}$ and some variable assignment g is a function which assigns to each wff either 0 or 1 under the following constraints.

> For any n-place predicate Π and any terms $\alpha_1 ... \alpha_n$,
> $\mathcal{V}^{\mathfrak{M},g}(\Pi\alpha_1 ... \alpha_n) = 1$ iff $\langle [\alpha_1]^{\mathfrak{M},g}, ... , [\alpha_n]^{\mathfrak{M},g} \rangle \in \mathcal{I}(\Pi)$

- For any wffs ϕ, ψ, and any variable α:

$\mathcal{V}^{\mathfrak{M},g}(\neg\phi)$	= 1 iff	$\mathcal{V}^{\mathfrak{M},g}(\phi) = 0$
$\mathcal{V}^{\mathfrak{M},g}(\phi \wedge \psi)$	= 1 iff	$\mathcal{V}^{\mathfrak{M},g}(\phi) = 1$ **and** $\mathcal{V}^{\mathfrak{M},g}(\psi) = 1$
$\mathcal{V}^{\mathfrak{M},g}(\phi \vee \psi)$	= 1 iff	$\mathcal{V}^{\mathfrak{M},g}(\phi) = 1$ **or** $\mathcal{V}^{\mathfrak{M},g}(\psi) = 1$
$\mathcal{V}^{\mathfrak{M},g}(\phi \rightarrow \psi)$	= 1 iff	$\mathcal{V}^{\mathfrak{M},g}(\phi) = 0$ **or** $\mathcal{V}^{\mathfrak{M},g}(\psi) = 1$
$\mathcal{V}^{\mathfrak{M},g}(\forall\alpha\phi)$	= 1 iff	**for every** $i \in \mathcal{D}$, $\mathcal{V}^{\mathfrak{M},g^{[i/\alpha]}}(\phi) = 1$
$\mathcal{V}^{\mathfrak{M},g}(\exists\alpha\phi)$	= 1 iff	**for at least one** $i \in \mathcal{D}$, $\mathcal{V}^{\mathfrak{M},g^{[i/\alpha]}}(\phi) = 1$

- We now define truth-in-a-model as follows.

> TRUTH-IN-A-MODEL: ϕ is *true in* a model $\mathfrak{M}$ iff $\mathcal{V}^{\mathfrak{M},g}(\phi) = 1$, for each variable assignment g for $\mathfrak{M}$.

Validity and Logical Consequence

- As usual, validity is defined as truth in all models and logical consequence is defined in terms of truth preservation.

> VALIDITY: ϕ is valid in $\mathfrak{L}$ iff ϕ is true in every model $\mathfrak{M}$.

> LOGICAL CONSEQUENCE: ϕ is a logical consequence of a set of wffs Γ in $\mathfrak{L}$ iff for every model $\mathfrak{M}$ and every variable assignment g for $\mathfrak{M}$, if $\mathcal{V}^{\mathfrak{M},g}(\gamma) = 1$ for every $\gamma \in \Gamma$, then $\mathcal{V}^{\mathfrak{M},g}(\phi) = 1$.

Existential Import

- It is standard in logic to treat the quantifiers in FOL as having existential import. That is, everything in the domain of quantification is assumed to exist. This means that the meaning of e.g. the existential quantifier can be paraphrased as follows:

$$\exists x Fx = \begin{cases} \text{At least one existing individual } x, x \text{ is } F \\ \text{There exists at least one } x \text{ such that } x \text{ is } F \\ \text{For some } x \text{ in the domain of existing things } D, x \text{ is } F \end{cases}$$

- So, existence statements in standard FOL are formulated using the existential quantifier rather than a predicate. For example, the statement that unicorns exist is formulated as a claim about the properties of the existing individuals/objects in the domain, i.e.

(1) 'Unicorns exist' $\rightsquigarrow \exists x \textbf{Unicorn} x$

- And, similarly, the statement that there are no unicorns is also formulated as a claim about the properties of the existing individuals or objects in the domain, i.e.

(2) 'Unicorns do not exist' $\rightsquigarrow \neg\exists x \textbf{Unicorn} x$

- One immediate problem with assuming that the existential quantifier has existential import is that it renders it impossible to capture the meaning of sentences such as (3) since the meaning of (3) is essentially equivalent to the seemingly contradictory (3a) and (3b).

(3) Some things do not exist.
  a. ~ At least one x (in the domain of existing things) does not exist.
  b. ~ There is at least one x (in the domain of existing things) such that x does not exist.

Relinquishing Existential Import

- Rather than accepting existential import, one could instead assume that the domain contains both existing and non-existing objects/individuals (including impossible objects such as round squares) and that the "existential" quantifier is simply an indicator of *proportion*.

- I.e. the statement 'something is F' is simply the statement that at least one x in the domain of (existing and non-existing individuals/objects) has the property F.

- When this approach is taken, existence is simply treated as a 1-place predicate, analogous to other 1-place predicates, e.g.

However, as Girle notes, this approach is not the orthodox approach in logic/philosophy and is strongly opposed by many.

QUESTION: What kinds of objections might one raise to the assumption that the existential quantifier *does not* have existential import?

(4) Some man runs. $\rightsquigarrow \exists x \textsf{\textbf{Run}} x$
(5) Some man exists. $\rightsquigarrow \exists x \textsf{\textbf{Exist}} x$

- One argument in favor of having existential quantifiers without existential import is that it might seem perfectly reasonable to have discussions/disagreements about what things exist.
- For example, suppose one encounters an individual who believes that there are round squares. It seems plausible to assume that one might respond to such an individual by saying:

(6) The things you are talking about, i.e. round squares, do not exist.

- In any event, once modal statements are taken into consideration, we are going to want to be able to express the possible existence of various objects/individuals. For example, (7) seems true.

(7) There might be another planet in the universe very similar to Venus.

- If the domain is assumed to consist of both (actually) existing and (actually) non-existing individuals/objects, capturing this is straightforward.
- For example, by giving up existential import, one could assume that the domain contains a planet very similar to Venus already and then interpret (7) as stating that this planet might exist. So, a paraphrase of (7) would be that given by (7a):

(7) a. ~ For some planet very similar to Venus x, it is possible that x exists.

To be fair, this can also be captured while assuming that the quantifiers have existential import, for example by assuming that the that the domain of existing things can vary from possible world to possible world (more on this later). When quantifiers are assumed to have existential import, but in this *world-relative* way, this is referred to as *actualist* quantification.

- So, we could translate (7) into quantified modal logic as follows:

(8) $\exists x \textsf{\textbf{similar-to-Venus}} x \Diamond \textsf{\textbf{exist}} x$

Weak Existential Import

- Another option is to assume that the quantifiers have *weak existential import.* This is the assumption that the domain of quantification consists of both *actually* and *merely possibly existing* objects/individuals (but not impossible objects such as round squares).

This is a referred to as *possibilist quantification*.

- In other words, on this view the quantifiers quantify over individuals/objects that *might* exist. This means that a formula such as (9) is translated as follows:

(9) $\exists x Fx \rightsquigarrow$ For some possibly existing x, x is F.

- However, one might worry that *possibilist quantification*, i.e. weak existential import, fails to capture the actual meaning of the existential and universal quantifiers—at least as they are used in English.

QUESTION: What precisely would the worry be here? (this worry would also apply to giving up existential import in the more general sense described above.)

Domains Across Possible Worlds

- When moving to quantified modal logic, the issues concerning quantificational domains become increasingly difficult.
- Setting aside the issue of existential import, in quantified modal logic, a question arises as to the nature of the quantificational domains.
- Generally speaking, there are three main options:
    - **Constant Domain Semantics**: The domain of quantification is *identical* for all possible worlds.

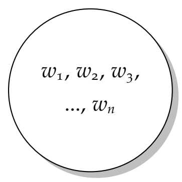


FIGURE: **Constant Domains** One quantificational domain for *all* possible worlds.

- **Distinct Domain Semantics**: The domain of quantification is *distinct* for each possible world.

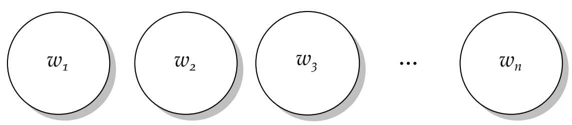


FIGURE: **Distinct Domains** One distinct quantificational domain for *each* possible world.

- **Variable Domain Semantics**: The domain of quantification varies from world to world. So, some worlds may overlap in domain with other worlds, and some worlds may have entirely distinct domains of quantification.

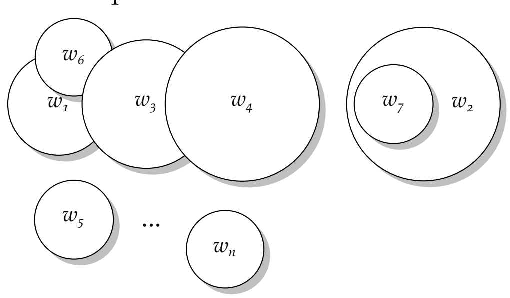


FIGURE: **Variable Domains** Multiple quantificational domains; some overlapping, some distinct.

- For example, in the illustration above, the quantificational domains of w_1, w_6, and w_3 all overlap, whereas the quantificational domain of w_5 is distinct from the domains of every other world.

- Furthermore, with a *variable* domain semantics, there are two prominent versions:
- One where for each accessible possible world, the domain expands (an ascending chain of domains):

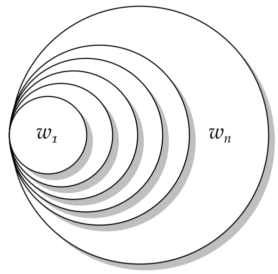


FIGURE: **Variable Ascending Domain** Domain increases with each accessible world.

- And one where for each accessible world, the domain contracts:

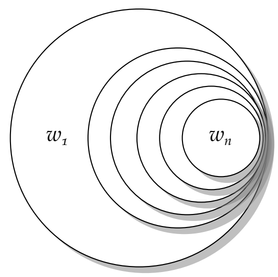


FIGURE: **Variable Descending Domain** Domain decreases with each accessible world

- For the remainder of these notes, we will assume a constant domain semantics (since this makes things significantly easier).

From FOL to QML

- When moving from FOL to quantified modal logic (QML), we need to make a number of changes to the logic (just as we did when we extended PL to MPL).

1. We enrich our lexicon by adding the two binary operators $\Box$ and $\Diamond$ to the lexicon.
2. We add the following syntactic/formation rule: If ϕ is a wff, then $\Box\phi$ and $\Diamond\phi$ are also formulas.
3. We add to our model a set of possible worlds $\mathcal{W}$ and a set of accessibility relations $\mathcal{R}$ where $\mathcal{R} \subseteq (\mathcal{W} \times \mathcal{W})$
4. We change our interpretation function $\mathcal{I}$ and our valuation function $\mathcal{V}$ to reflect that predicate extensions are world-relative, so that the truth value of wffs can change depending on the world of evaluation, i.e.

Specifically, the interpretation function $\mathcal{I}$ maps predicates to extensions at possible worlds.

A Valuation Function $\mathcal{V}$ for QML:
For any n-place predicate Π and any terms $\alpha_1 ... \alpha_n$,
$\mathcal{V}^{\mathfrak{M},g,w}(\Pi\alpha_1 ... \alpha_n) = 1$ iff $\langle [\alpha_1]^{\mathfrak{M},g}, ... , [\alpha_n]^{\mathfrak{M},g} \rangle \in \mathcal{I}(\Pi, w)$

5. We give the standard semantic interpretation of $\Box$ and $\Diamond$, i.e.
   $\mathcal{V}_{\mathfrak{M}}(\Box\phi, w) = 1$ iff for all $w' \in \mathcal{W}$, if $w\mathcal{R}w'$, then $\mathcal{V}_{\mathfrak{M}}(\phi, w') = 1$
   $\mathcal{V}_{\mathfrak{M}}(\Diamond\phi, w) = 1$ iff for some $w' \in \mathcal{W}$, $w\mathcal{R}w'$ and $\mathcal{V}_{\mathfrak{M}}(\phi, w') = 1$

The Barcan Formula and Its Converse

- One interesting consequence of assuming a constant domain semantics is an equivalence now commonly known as the Barcan Formula (and the Converse Barcan Formula).
- The Barcan formula is stated in (10) which is equivalent to (11):

These equivalences are named after the logician Ruth Barcan Marcus who proved them in 1946/1947.

(10) $\forall x \Box Fx \rightarrow \Box \forall x Fx$
(11) $\Diamond \exists x Fx \rightarrow \exists x \Diamond Fx$

- If the Barcan Formula is assumed as an axiom, it implies that there are no merely possible individuals/objects. That is, if it is possible that an individual (with a certain property exists), then that individual *actually* exists.

This thesis is also known as *actualism*.

- In other words, if the Barcan formula is assumed as an axiom, it entails that the domain of the quantifiers cannot *expand* across accessible possible worlds.
- The Converse Barcan formula is stated in (12) which is equivalent to (13):

EXERCISE: Whether the Barcan Formula is valid remains controversial. If you think about what the BF means, can you think of a reason why it might be invalid?

(12) $\Box \forall x Fx \rightarrow \forall x \Box Fx$
(13) $\exists x \Diamond Fx \rightarrow \Diamond \exists x Fx$

- If the Converse Barcan formula is assumed as an axiom in a modal system, this implies that the domains cannot shrink across accessible possible worlds.
- As should be fairly clear, if the quantificational domain is assumed to be *constant*, both the BF and the CBF are going to be valid.

Modalities *De Dicto* and *De Re*

- It is standard to distinguish between to distinct types of modality, namely *de dicto* and *de re*. Roughly speaking, these can be explained as follows:
  - DE DICTO MODALITY: When necessity or possibility is attributed to a *proposition*.
  - DE RE MODALITY: When an individual or object is said to possibly or necessarily have a certain property.

This could be considered a misleading way of putting things for various reasons, but for a first rough gloss this will do.

- Formally, the distinction between *de dicto* and *de re* modality can be captured in terms of scope. Again, simplifying somewhat, if a modal is taking scope over a closed sentence, the modality is *de dicto*, cf. (14). By contrast, if the modal is taking scope over an open sentence (where the variable in that open sentence is bound from the outside), the modality is *de re*, cf. (15).

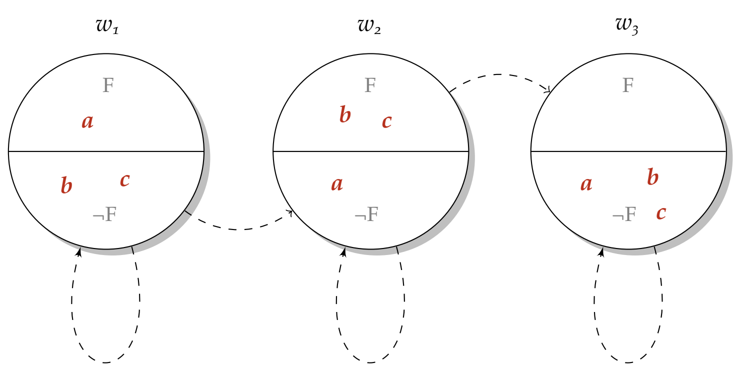
(14) $\Box\exists x Fx$ (*de dicto* modality)
(15) $\exists x \Box Fx$ (*de re* modality)

The *de dicto*/*de re* distinction has been subject of much discussion in philosophy. Quine, for example, who is generally skeptical of quantified modal logic, has raised a worry about quantifying into modal environments. According to Quine, it is borderline non-sensical to attribute properties to objects necessarily when this attribution fails to take into account how the object is described (which is precisely what can happen in de re attributions).

- In (14), the sentence '$\exists x Fx$' is claimed to be necessary, i.e. the proposition expressed by that sentence is claimed to be true in every accessible world. By contrast, what is claimed in (15) is that for some individual x, x has the property F necessarily, i.e. in every accessible world x is F.
- In other words, while (14) is true as long as in every accessible world some individual x is F, (15) is only true if some specific individual x is F in every accessible world.

The Interpretation of Quantified Modal Sentences

- In order to explicate when (and why) quantified modal sentences are true/false relative to various models, let's consider a simple setup:
- Suppose we have only three possible worlds w_1, w_2, and w_3 with a constant and finite domain consisting of only three individuals, a, b, and c.



- Moreover, we assume a simple reflexive T-model setup where w_1 has access to w_2, and w_2 has access to w_3. There are no other accessibility relations.

Exercises

- Let's now consider some quantified modal statements and their status relative to this model.

(16) $\mathcal{V}^{\mathfrak{M},g}(\Diamond\exists xFx, w_1) = ?$

(17) $\mathcal{V}^{\mathfrak{M},g}(\Box\exists xFx, w_1) = ?$

(18) $\mathcal{V}^{\mathfrak{M},g}(\exists x\Diamond Fx, w_3) = ?$

(19) $\mathcal{V}^{\mathfrak{M},g}(\exists x\Box Fx, w_2) = ?$

(20) $\mathcal{V}^{\mathfrak{M},g}(\Diamond\forall xFx, w_1) = ?$

(21) $\mathcal{V}^{\mathfrak{M},g}(\Box\forall xFx, w_3) = ?$

(22) $\mathcal{V}^{\mathfrak{M},g}(\forall x\Diamond Fx, w_2) = ?$

(23) $\mathcal{V}^{\mathfrak{M},g}(\forall x\Box Fx, w_1) = ?$

- A couple of slightly harder ones:

(24) $\mathcal{V}^{\mathfrak{M},g}(\Diamond\forall x(Fx \to \Diamond\exists y\neg Fy), w_1) = ?$

(25) $\mathcal{V}^{\mathfrak{M},g}(\Box\forall x(Fx \to \exists y\Diamond Fy), w_1) = ?$

Logic 2: Modal Logics – Week 8

Individual Constants

- In the previous lecture, we focused on the introduction of quantifiers into modal logic and we discussed issues about existential import and the nature of the quantificational domains. Now, we turn our attention to singular terms—starting with individual constants.
- So, remember, in standard predicate logic, our lexicon includes a set of individual constants a, b, c, etc. The meaning/extension of an individual constant is determined by the assignment function which maps these to individuals in the domain. So, one interpretation function $\mathcal{I}$ might provide the following mapping.

$$\mathcal{I}: \begin{bmatrix} a & \longrightarrow & \text{Pluto} \\ b & \longrightarrow & \text{The Dugald Stewart Building} \\ c & \longrightarrow & \text{Theresa May} \\ \vdots & & \vdots \end{bmatrix}$$

- Individual constants thus contrast variables which are not interpreted by $\mathcal{I}$, but by a variable assignment g instead.
- So, essentially, we are going to think of individual constants as names and we are going to more or less ignore variables for now

Although, as I mentioned in class, one might think of variables as pronouns.

Numerical Identity

- To add expressive power to our logical system, we are going to add numerical identity—expressed by the connective '='. So, we add the following syntactic rule: If α and β are singular terms, then '$\alpha = \beta$' is a formula.
- With '=' in our inventory, we can now express statements of numerical identity, e.g.

  (1) Cicero is Tully.
  (2) Superman is Clark Kent.
  (3) Hesperus is Hesperus.

- In our formal language, identity statements look as follows:

  (4) $a = b$
  (5) $c = c$

- Without numerical identity in predicate logic, it is impossible to formulate a range of claims, e.g. those below.

  (6) There is exactly one president
  $\exists x(\textbf{President}x \wedge \forall y(\textbf{President}y \rightarrow x = y))$

  (7) There are at most two frogs.
  $\exists x \exists y((\textbf{Frog}x \wedge \textbf{Frog}y) \rightarrow \forall z(\textbf{Frog}z \rightarrow (z = x \vee z = y)))$

Identity vs. Predication

- In English, numerical identity is expressed using the word 'is' (or possibly the more cumbersome 'is identical to'), but it is important to distinguish between different kinds of uses of 'is', namely the 'is' of predication vs. the 'is' of identity.
- In the examples above, two individuals are asserted to be numerically identical. However, in the examples below, this is not the case.

(8) Frank is tired.
(9) Louise is a professor.

- That is, in (8), Frank is not asserted to be identical to a property and in (9), Louise is not asserted to be identical to a professor. Rather, in these sentences, a property is predicated of Frank and Louise.
- It just so happens, that in English, the word 'is' is used to predicate properties of individuals as well. And, when the property does not have a corresponding verbal predicate (e.g. 'laughs', 'snores', or 'tall'), one must use the somewhat cumbersome 'is a' instead (as in 'is a professor', 'is a master of disguise', or 'is a genius').

Similarly, in (9) the word 'a' is not a quantifier expression either.

Existential Import and Logical Principles

- There are a range of logical inferences that are going to be valid when it comes to individual constants, e.g.

$$\frac{Fa}{\exists x Fx}$$

- However, the correct translation of these formulas will depend on assumptions about the domain of quantification. For example,
  - If our semantics has *actualist* quantification, the conclusion of this inference is that some *existing* individual is F.
  - If our semantics has *possibilist* quantification, the conclusion of this inference is merely that some existing or possibly existing individual is F.
- In standard predicate logic, the following is also a logical truth:

(10) $\exists x(x = a)$

- However, what the correct translation of this sentence is depends on assumptions about the nature of the quantificational domain. If the quantifiers have existential import (and if, say, a is a name for Andy Murray), then the meaning of the sentence above is simply:

(11) Andy Murray exists.

- Too see that (10) is a logical truth, simply consider the negation.

(12) $\neg\exists x(x = a)$

- From (12), we can infer $\forall x\neg(x = a)$ and by universal instantiation, we reach the absurd conclusion $\neg(a = a)$.
- So, since it seems a bit strange to think that (11) is a logical truth, this might be a reason to consider giving up existential import. If we give up existential import, we can maintain that (10) is a logical truth without having to accept that it has the meaning in (11).
- If our semantics has *possibilist* quantification, the right translation of (10) is (13). Intuitively, accepting this as a logical truth is less controversial.

(13) Andy Murray might exist.

- Also, the negation of 13 would be (14), which presumably should be rejected.

(14) It's impossible for Andy Murray to exist.

Descriptive Singular Terms

- In standard predicate logic, individual constants and individual variables (both singular terms) have no descriptive content. The "meaning" of these expressions is simply an individual, the individual assigned by the interpretation function (for constants) or the variable assignment (for variables).
- However, there is a class of expressions in natural language that very much seem like singular terms but also appear to have substantial descriptive content, namely definite descriptions:

(15) The president of the United States
(16) The king of France
(17) The table

- There are a number of different ways of trying to introduce definite descriptions into predicate logic. Girle considers the following two options:
  - SINGULAR DEFINITE DESCRIPTION OPERATOR
    One option is to introduce a dedicated singular quantifier expression, ιx, which combines with formulas just as other quantifiers do, i.e.

$$\iota x Fx \quad | \quad \iota x(Fx \wedge Gx)$$

- However, syntactically, this kind expression can only occur in the same positions as singular terms, i.e.

$$\textbf{Frog}(x) \quad | \quad \textbf{Frog}(a) \quad | \quad \textbf{Frog}(\iota x F x)$$

- The latter sentence is then supposed to be translated as follows:

(18) The unique individual who is F is a frog.

- Girle does not say much more than this about this way of going for definite descriptions and unfortunately this leaves open a number of rather complicated questions.

- THE QUANTIFICATIONAL THEORY
  Another option is Russell's proposed analysis where definite descriptions are given a *contextual* definition. So, as in the previous case, a definite description operator is introduced, viz. ιx, which combines with open formulas.
- So, again, we can write formulas such as:

$$\textbf{Frog}(\iota x F x)$$

- However, such a sentence is really shorthand for something more complex, namely the formula in (19):

(19) $\exists x(Fx \wedge \forall y(Fy \rightarrow y = x) \wedge \textbf{Frog}x)$

- Russell's analysis of descriptions avoids many of the problems for the previous approach and it has a range of additional advantages. However, Russell's analysis also has a number of problems and while it was once true that this is, as Girle puts it, the 'most widely accepted' view of definite descriptions, that's not obviously true anymore.

Existential Import: Options

- We've already seen that when it comes to the quantifiers of predicate logic, a choice has to be made as to whether these have existential import or not – i.e. does the domain consist of only existing individuals or does it include non-existent individuals. We are faced with a similar choice for the individual constants.

- In standard predicate logic, there are no constants in the language without an extension (i.e. without a reference). This means the existential commitments incurred by individual constants depends on antecedent choices about the nature of the domain.

- For example, if we assume that the domain consists of only existing individuals (*actualist* quantification), the individual constants will then have existential import.

- And if we assume that the domain consists of both existing and merely possibly existing individuals, the individual constans will not have existential import.

- There is, however, an intermediate option: Let the domain consists of both existing and non-existing individuals, but assume that the quantifiers only range over a subset of this domain, namely the subset of existing individuals. This is referred to as *Free Logic*.

Free Logic

- In Free Logic, the quantifiers *have* existential import, but the individual constants do not.
- So, an inference from Fa to $\exists xFx$ is therefore invalid.
- However, the formula below is a logical truth:

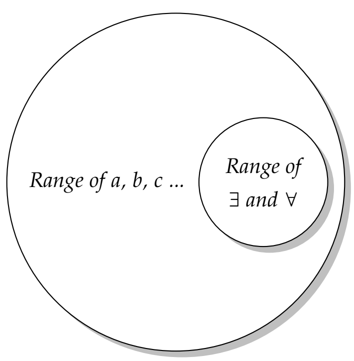
$$(\exists x(x = a) \wedge Fa) \rightarrow \exists xFx$$

- So, in free logic there is no problem with statements of non-existence involving names (let a be name of Vulcan):

(20) Vulcan does not exist.
(21) $\neg\exists x(x = a)$

- In other words, in Free Logic, names might fail to refer.
- We can visualize the nature of the domain in Free Logic as follows:

DOMAIN IN FREE LOGIC



- Free Logic is particularly well suited to deal with languages containing names for fictional entities, e.g. ‘Sherlock Holmes’, ‘Batman’, etc.
- For example, one can easily represent the truth or falsity of sentences such as the following:

(22) Sherlock Holmes lives on 221B Baker Street.
(23) Batman is Bruce Wayne
(24) Superman is faster than every airplane.

- However, Free Logic also has a range of somewhat strange consequences: For example, if definite descriptions are treated as Russell suggested, then while e.g. the sentence in (24) is true, the corresponding sentence in (25) is going to be false.

(25) The famous superhero born on Krypton is faster than an airplane.

- This seems somewhat odd given that the description in (25) is supposed to pick out Superman.
- Girle suggest that one way to avoid this problem would be to introduce a new quantifier with the meaning 'exactly one' that ranges over the entire domain and does not have existential import. This would then be the quantifier that is used to capture the meaning of 'the *F* is *G*'.
- But it's not clear why we should be so worried about definite descriptions failing to pick out the right things if we are not also concerned about statements such as those below.

(26) Superman defeated at least one other superhero in a recent movie.
(27) Every superhero wears a special suit.

Two New Quantifiers

- Yet another option is to introduce *two* new quantifiers, namely Σ (at least one) and Π (every) where neither of these have existential import.
- If we did this, we would be able to capture the meaning of (28) below (which we were not able to previously).

(28) At least one thing does not exist.
(29) $\Sigma x \forall y (x \neq y)$

- We can then use these new quantifiers to rescue the Russellian theory of descriptions, because we can simply assume that these quantifiers are used for descriptions (when needed).
- Moreover, we can now capture the meaning of the sentences in (26) and (27).
- One consequence of this view, however, is that natural language quantifiers are genuinely ambiguous. On one interpretation, they have existential import and on the other interpretation they do not. But it is not clear that there is any compelling evidence that natural language quantifiers are ambiguous in this way.
- Again, one might be tempted to think that we should either keep existential import or simply give it up so that the standard quantifiers range over even non-existent (but possibly existing) individuals — i.e. *possibilism*.

Individual Constants in Modal Environments

- Let's now consider how to evaluate formulas in MPL involving individual constants, e.g. (30) and (31)

(30) $\Diamond Fa$
(31) $\Box Fa$

- Consider (30). Whether this formula is true depends on (a) our assumptions about the domain, and (b) our assumptions about domains across worlds.

EXERCISE: Work out the truth conditions for (31) given these options.

(AC) **Actualist Quantification + Constant Domains**
(30) is going to be true only if there is an accessible world v such that a (who exists) is F at v.

(PC) **Possibilist Quantification + Constant Domains**
(30) is going to be true only if there is an accessible world v such that a (who *might* exist) is F at v.

(AV) **Actualist Quantification with Variable Domains**
(30) is going to be true only if there is an accessible world v such that a exists at v and a is F at v.

(PV) **Possibilist Quantification with Variable Domains**
(30) is going to be true only if there is an accessible world v such that a is in the domain of v and a is F at v.

- (AC): The assumptions that yield the least complexity is actualist quantification and constant domains, i.e. everything in the domain exists and the quantifiers range over the same domain in every world.

  This makes life easier in many ways, but it also means that it becomes difficult to capture the truth of certain sentences, e.g. (32) below.

(32) Vulcan might exist.

- (PC): Moving to possibilist quantification with constant domains fixes this problem. The sentences above are easily captured if we assume that the domain includes possibly existing objects.
- But once we accept possibilist quantification, we are assuming that the quantifiers range over *merely possibly existing* individuals. As discussed earlier, this might seem dubious.
- (AV): Assuming actualist quantification but variable domains might seem the most natural option. This way we retain existential import (our quantifiers only quantify over existing individuals), but we allow for the possibility that certain objects/individuals (that do not exist in the actual world) may exist in other worlds.

- However, our semantics now gets vastly more complicated because in order to check for validity, we now need to consider not only every possible assignment of extensions to our terms, but also every possible variation in the domains.
- (PV): This is the most expressively powerful but also most complex option. Again, with this option, we would be able to capture sentences such as (32) above. However, it is not entirely clear why we for every world we need the domain to include both existing and non-existing objects.

Names and Rigid Designation

- It has been implicitly assumed in all of the above that names are *rigid designators*. That is, a name a refers to the same individual in every possible world (assuming the object exists).
- Using natural language as a guide, this seems like a sensible assumption about names, cf. Kripke's many arguments in *Naming and Necessity*.
- If it is assumed that names are rigid, then it follows that identity statements are necessary. That is, sentences such as those below are necessary truths.

  (33) Hesperus is Phosphorus.
  (34) Agatha Christie is Mary Westmacott.

- The necessity of identity is a substantial philosophical assumption and one might have reasons to think that identity is contingent. However, this would require that names are not treated as rigid designators, i.e. it would need to be possible for a constant a to have different extensions relative to different possible worlds.
- The most natural way to do this is to treat "constants" similar to the way that predicates are treated, namely as having world-variable extensions. So, instead of thinking of the interpretation function $\mathcal{I}$ simply as mapping constants to individuals in the domain, we should think of $\mathcal{I}$ as mapping pairs of constants and worlds to extensions (where this extension might change depending on the world).
- Obviously, assuming that names are non-rigid greatly complicates the logic, but whether names are rigid or not is an empirical issue that should not be decided based on convenience.
- That said, the arguments for assumption that names are rigid are quite compelling and has convinced most researchers in philosophy of language, logic, and linguistics.

Exercises

- Construct countermodels for the following formulas:

1. $\nvDash_{T} (\Diamond Fa \wedge \Diamond Ga) \rightarrow \Diamond(Fa \wedge Ga)$
2. $\nvDash_{S4} (\Diamond \exists x Fx \wedge \Box \exists y Gy) \rightarrow \Box(\exists x Gx \wedge \exists y Fy)$
3. $\nvDash_{T} (\Box Fa \rightarrow \exists y \Diamond Fy) \wedge \Box(\exists y \Diamond Fy \rightarrow Fa)$

Logic 2: Modal Logics – Week 8

Epistemic and Doxastic Logics

- On the *epistemic* interpretation of modal logic, '$\Box\phi$' is interpreted as 'it is known that ϕ'.
- On a *doxastic* interpretation of modal logic, '$\Box\phi$' is interpreted as 'it is believed that ϕ'.
- In other words, these are the logics for *knowledge* and *belief*.

Agents and Operators

- In epistemic and doxastic logic, it is customary to use subscripts on the on modal operators to indicate the agent in question. Moreover, standardly, the $\Box$ is written as K in epistemic logic and B in doxastic logic.
- So, the sentences below would be formalized as follows:

(1) Betty knows that ϕ. $\rightsquigarrow$ $K_b\phi$
(2) Andy believes that ϕ. $\rightsquigarrow$ $B_a\phi$

- If we are considering a multi-agent epistemic or doxastic logic, we will need a different modal operator for each agent. For example, if we're considering a logic with four agents, namely Andy, Betty, Carly, and Deryn, we will need K_a, K_b, K_c, and K_d.

Duals

- As in standard modal logics, the $\Box$ has a dual, namely $\Diamond$.
- In epistemic logic, the dual of K_x is P_x and in doxastic logic the dual of B_x is C_x. These are defined as follows:

$$P_x =_{def} \neg K_x \neg$$
$$C_x =_{def} \neg B_x \neg$$

- So, a formula such as (3) below, would be the translation of e.g. (3a), (3b), or (3c). Same for (4) and (4a), (4b), and (4c).

(3) $P_a\phi$
   a. It's not the case that Andy knows that not-ϕ.
   b. Andy doesn't know that not-ϕ.
   c. ϕ is compatible with Andy's knowledge.

(4) $C_a\phi$
   a. It's not the case that Andy believes that not-ϕ.
   b. Andy doesn't believe that not-ϕ.
   c. ϕ is compatible with Andy's beliefs.

Epistemic Logic: S4

- The main difference between single modality logics (e.g. *alethic* modality logic) and multi-modal logics (i.e. a multiple agent logic) is that the models will contain an accessibility relation for each agent.
- In epistemic logic, we use E for the accessibility relation and, again, this will be indexed to an agent: E_a, E_b, E_c, ...

Tree Rules for Epistemic Logic S4

- The standard tree rules for propositional logic, **PTr**, remain unchanged, but we add the following rules for S4 epistemic logic:

KPN-rules

$\neg K_x\phi \quad (w) \quad \checkmark$
$\vdots$
$P_x\neg\phi \quad (w)$ (**KPN**)

$\neg P_x\phi \quad (w) \quad \checkmark$
$\vdots$
$K_x\neg\phi \quad (w)$ (**KPN**)

$K_x\neg\phi \quad (w) \quad \checkmark$
$\vdots$
$\neg P_x\phi \quad (w)$ (**KPN**)

$P_x\neg\phi \quad (w) \quad \checkmark$
$\vdots$
$\neg K_x\phi \quad (w)$ (**KPN**)

PR

$P_x\phi \quad (w) \quad \checkmark$
$\vdots$
wE_xv
$\phi \quad (v)$ (**PR**)
↑
where v is new to path.

KR

$K_x\phi \quad (w) \quad \checkmark$
wE_xv
$\vdots$
$\phi \quad (v)$ (**KR**)
↑
for any v.

KT

$K_x\phi \quad (w) \quad \checkmark$
$\vdots$
$\phi \quad (w)$ (**KT**)

KKR

$K_x\phi \quad (w) \quad \checkmark$
wE_xv
$\vdots$
$K_x\phi \quad (v)$ (**KKR**)
↑
for any v.

- The **KPN-rules** are the correlate in epistemic logic of the standard modal negation rules (MN-rules).

- **PR** says that if x doesn't know that not-ϕ, then there is a world accessible (i.e. a world compatible with x's knowledge) such that ϕ is true there.

- Similarly, **KR** says that for ϕ to be known by x, ϕ must be true in every accessible world (i.e. every world compatible with x's knowledge).

- **KT** is the correlate of the **T**-rule in standard modal logic. It simply says that for ϕ to be known by x, ϕ must be true. This also sometimes referred to as *factivity* or *veridicality*.

So, instead of this rule, we could just have **(Refl).**

- **KKR** is the specific S4 rule. This guarantees that if a knows that ϕ, then a knows that he knows that ϕ.

So instead of this rule, we could just add (**Trans**). To see this, just consider what a model would have to look like to make $K_x\phi$ true at w, but false at v where wE_xv. This is not possible if the accessibility relation is transitive.

Multi-Agent Rules

- The rules above are all single agent rules, but we also add the following multi-agent rule referred to as the *Transmission of Knowledge Rule*.

TrKR-rule $\quad K_xK_y\phi \quad (w) \quad \checkmark$

$\vdots$

$K_x\phi \quad (w) \qquad$ (**TrKR**)

> IN CLASS EXERCISES:
> Using semantic trees, show that the formulas below are valid:
>
> 1. $\vDash_{S4} K_a\neg K_a(p \to p) \to K_a(p \to K_a p)$
> 2. $\vDash_{S4} (K_a(p \to q) \wedge K_a p) \to \neg P_a K_a \neg q$
> 3. $\vDash_{S4} (K_a K_b(p \to q) \wedge K_a \neg q) \to K_a(p \vee r)$
> 4. $\vDash_{S4} P_a P_a \phi \to P_a \phi$

- From an axiomatic point of view, the S4 version of epistemic logic that we are currently considering includes the following axioms:

(*K1*) $K_x(\phi \to \psi) \to (K_x\phi \to K_x\psi)$

Also known as *Distribution*.

($K2$) $K_x\phi \to \phi$

Also known as *Factivity* or *Veridicality*.

($K3$) $K_x\phi \to K_xK_x\phi$

Also known as *Positive Introspection* or the *KK-Principle*.

($K4$) $K_xK_y\phi \to K_x\phi$

Also known as *Transmissibility of Knowledge*

- Of these axioms, only ($K2$) is generally considered uncontroversial. We will discuss the consequences of these axioms in turn.

(The Problem of) Omniscience

- Subtle questions arise when using modal logics to model knowledge as regards what exactly we modelling. For example, are we trying to model the structure of knowledge in an *ideal* (and non-finite) knower? Or are we trying to model the structure of human knowledge?
- If the latter, how much should we assume that a human agent knows? Does the agent know the laws of logic? The laws of epistemic logic? Any logic?
- Epistemic logics generally imply a certain kind of omniscience on the part of its agents, but to understand which kinds, let's consider a couple of different definitions:
  - Logical Omniscience
  - Deductive Omniscience
  - Factual Omniscience
- *Logical Omniscience*: Following Girle, let's distinguish between strong and weak logical omniscience.
  - If an agent is represented as automatically knowing every logical truth (including modal logical truths), we'll say that the agent is **strongly logically omniscient** (SLOT).
  - If an agent is represented as automatically knowing every logical truth of propositional logic or first order logic, we'll say that the agent is **weakly logically omniscient** (WLOT).

As Girle points out, Descartes seemed to be an advocate of WLOT.

- *Deductive Omniscience*
  - If an agent is represented as knowing the logical consequences of every known proposition, we'll say that the agent is **deductively omniscient** (DOT).
- *Factual Omniscience*
  - If an agent is represented as knowing every true proposition, i.e. for every proposition p, if the agent knows whether p or $\neg p$, we'll say that the agent is **factually omniscient** (FOT).
- Girle considers a range of non-normal modal logics, but here we'll focus on the normal modal logics T, S4, and S5.

Logical Omniscience

- In standard axiomatic approaches to modal logic, there is a rule of necessitation. This rule says the following:

Here Δ is a variable ranging over the normal modal logics.

$$\vdash_\Delta \phi \Rightarrow \vdash_\Delta \Box\phi$$

- That is, if there is a proof (from the empty set of premises) of a formula ϕ, then there is a proof of $\Box\phi$.
- From a semantic point of view, this makes perfect sense. If there is a proof of ϕ from the empty set of premises, this means that ϕ is a logical truth, i.e. ϕ cannot be false. If ϕ is a logical truth, then ϕ holds in every possible world. Hence, $\Box\phi$ is true.

What we should say here is really this: If the logic is *sound* (which all the normal modal logics are), it follows that:

$$\vdash_\Delta \phi \Rightarrow \vDash_\Delta \phi$$

- So, we also have:

$$\vDash_\Delta \phi \Rightarrow \vDash_\Delta \Box\phi$$

- Consider, for example, the rule of **necessity introduction** (NI) for semantic trees. This is a general rule applying to T, S4, and S5. Since K_x is simply the epistemic correlate of $\Box$, and since it seems natural to assume that anything that is provable from the empty set is necessary, there is a corresponding introduction rule for K_x.
- As a result, we get the following in T, S4, and S5. We refer to this as **epistemic necessitation**.

$$\vdash_\Delta \phi \Rightarrow \vdash_\Delta K_x\phi$$

- In other words, in all of the normal modal logics, agents are going to be *strongly* omniscient.
- One way to avoid this result, is to give up necessitation in place of something weaker, e.g. **weak necessitation**:

$$\vdash_{PL} \phi \Rightarrow \vdash_\Delta \Box\phi$$

- This rule says that if a formula can be proven from the empty set in PL, then it is necessary. If weak necessitation is adopted, then when transposing it into epistemic logic, it would have the more limited consequence that only every logical truth *in PL* is known.
- However, it would also mean that there are no modal logical truths which seems bizarre.

Deductive Omniscience

- The weakest of the normal modal logics is K and every other normal modal logic is an extension of K.

This is therefore known as the K-axiom.

- In K (and hence every other normal modal logic) the following is an axiom.

$$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

- In epistemic logics that are simply extensions of one of the normal modal logics, we therefore get the corresponding (*K1*) axiom.

(*K1*) $K_x(\phi \to \psi) \to (K_x\phi \to K_x\psi)$

- Now, this axiom, the K-axiom (or the distribution axiom), is not itself going to lead to deductive omniscience. Deductive omniscience requires the following theorem.

$$\vdash_\Delta (\phi \to \psi) \to (K_x\phi \to K_x\psi)$$

Deductive Omniscience

- But all we can get from the K-axiom on its own is:

$$\vdash_\Delta K_x(\phi \to \psi) \to (K_x\phi \to K_x\psi)$$

- And this doesn't seem obviously unreasonable.
- But when distribution is combined with strong omniscience, we get deductive omniscience as a result.

  Proof. Suppose that $\phi \vdash_\Delta \psi$. If so, then given logical omniscience it follows that $\vdash_\Delta K_x(\phi \to \psi)$. And then using the distribution axiom, it follows that $\vdash_\Delta K_x\phi \to K_x\psi$. So, if $\phi \vdash_\Delta \psi$, it follows that $K_x\phi \vdash_\Delta K_x\psi$. QED

- In conclusion, in T, S4, and S5, agents are both strongly omniscient and deductively omniscient.
- Let's now consider some of the consequences of the individual logical systems – starting with S4.

Knowing That You Know

- In S4, agents are strongly omniscient and deductively omniscient. However, remember, in S4, we also have the following axiom:

(*K3*) $K_x\phi \to K_xK_x\phi$

Also known as *Positive Introspection* or the *KK-Principle*.

- So, in S4, for any proposition ϕ, if an agent knows that ϕ, the agent will know that she knows that ϕ.

- The S4-axiom is controversial in epistemic logic and there are many sophisticated arguments purporting to show that it fails.

One prominent example is Timothy Williamson's arguments in 'Knowledge and Its Limits' (Oxford, 2000).

- Here is what might seem a very simple argument against the S4-axiom:

> If the-S4 axiom governs knowledge, then everyone should not only know that it is true, they should also know that they know that it is true, know that they know that they know that it is true, and so on.
>
> So, suppose that it is an axiom governing the nature of knowledge. If so, we know (by necessitation) that it's true. Moreover, we should know that we know that it is true. But since certain people deny that it is true, it is clear that they do not know that they know that it is true. So, it must be false.

- This argument basically says that if the S4 axiom holds, it should not be possible to deny that it holds.
- Yet, one might think that the immediate problem that this argument reveals is not with the *S*4-axiom itself, but rather necessitation.

Limits on Knowledge in S4

- In S4 there are *some* limits on agents' knowledge. For example, in S4, it is possible for an agent to be mistaken about what she believes.
- That is, the following are consistent formulas:

(5) $\neg\phi \wedge \neg K_a \neg K_a \phi$
(6) $\neg K_a \phi \wedge \neg K_a \neg K_a \phi$

- In other words, the S4-agent can be ignorant about what she knows and what she doesn't know.

Knowing That You Don't Know

- The axiom characterizing S5 is the following:

$$\Diamond\phi \rightarrow \Box\Diamond\phi$$

- So, for an S5 epistemic logic, we get the corresponding axiom:

(*K*5) $P_x\phi \rightarrow K_x P_x \phi$

- The equivalent of which is:

$$\neg K_x \neg \phi \rightarrow K_x \neg K_x \neg \phi$$

- From this axiom, the following are going to be theorems:

(*T1*) $\neg \phi \rightarrow K_x \neg K_x \phi$ — The Platonic Principle

(*T2*) $\neg K_x \phi \rightarrow K_x \neg K_x \phi$ — Negative Introspection

Re: (*T1*). Let ϕ be $\neg p$. So, suppose $\neg\neg p$ is true. That is, p is true. If so, $\neg p$ is false. If $\neg p$ is false, it cannot be known. So, it follows that $\neg K_a \neg p$. But then by (*K5*), it follows that $K_a \neg K_a \neg p$, i.e. $K_a \neg K_a \phi$.

- In other words, in S5 epistemic logic, every agent knows exactly how ignorant they are. For any proposition ϕ, if ϕ is false, they know that they don't know that ϕ is true.
- This does not quite amount to factual omniscience, since it is possible for an agent to know that they don't know that ϕ, but also to know that they don't know that $\neg \phi$.
- But, the S5 agent is nevertheless an incredibly epistemically powerful agent: She is logically omniscient, deductively omniscient, and knows for any false proposition that she does not know that proposition.

Girle writes that there is only possible S5 knower, namely an omniscient god. But one might think that an omniscient god would also be factually omniscient.

Knowledge in T

- In the modal logic T, we have the axioms (*K1*), (*K2*), and epistemic necessitation. So, agents in T are strongly omniscient and deductively omniscient.
- However, since neither the S4-axiom nor the S5-axiom holds in T, the T-agent is much more restricted in her knowledge. For example, it is possible for the T-agent to know that ϕ without knowing that she knows that ϕ (failure of the S4-axiom). It's also possible for ϕ to be false without the agent knowing that she doesn't know that ϕ (failure of 5-axiom).
- But despite the lack of the S4-axiom in T, epistemic necessitation entails that any logical truth is known and since knowledge of logical truths are theorems of T, it follows that knowledge of logical truths is itself knowledge.
- In other words, even in T, the KK-thesis holds in a restricted way:
  - Given necessitation:

$$\vdash_T \phi \Rightarrow \vdash_T K_x \phi$$

  - So, it follows that if ϕ is a theorem in T, then $\vdash_T K_x K_x \phi$

Logic 2: Modal Logics – Week 10

Natural Language Conditionals

- The analysis of natural language conditionals as material implications is unsatisfactory for a variety of reasons. Specifically, this analysis validates a number of inferences that seem dubious.

Girle refers to the converse of this principle, namely $(\phi \wedge \neg\psi) \rightarrow \neg(\phi \rightarrow \psi)$, as Jackson's *uncontested principle*.

- NEGATED CONDITIONAL TO CONJUNCTION

$$\neg(\phi \rightarrow \psi) \rightarrow (\phi \wedge \neg\psi)$$

- The formula above is valid in standard propositional logic. So, if material implication is the right analysis of conditionals in English, this means that the following inference should be valid:

(1) It's not true that if there will be a minor earthquake in Edinburgh tomorrow, the Dugald Stewart Building will collapse.

Therefore, there will be a minor earthquake in Edinburgh tomorrow and the Dugald Stewart Building will not collapse.

- ANTECEDENT STRENGTHENING

This also sometimes referred to as *augmentation* and (confusingly) *weakening*.

$$(\phi \rightarrow \psi) \rightarrow ((\phi \wedge \chi) \rightarrow \psi)$$

- This formula is also valid in standard PL, so this means that the following inference should be valid:

(2) If Jack flips the switch, the light comes on.

Therefore, if Jack flips the switch and turns off the electricity, the light comes on.

- CONTRAPOSITION

$$(\phi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi)$$

- From the validity of contraposition, it follows that the inference below should be valid:

(3) If Herman moves in, Ernie will not move out.

Therefore, if Ernie moves out, then Herman will not move in.

To see why this inference is problematic, just suppose Ernie really wants to live in the same house as Herman, but that Herman has no specific desire to live in the same house as Ernie — Herman just wants to live in that particular house.

· PARADOXES OF MATERIAL IMPLICATION

$$\phi \rightarrow (\psi \rightarrow \phi)$$
$$\neg\phi \rightarrow (\phi \rightarrow \psi)$$

- The problem commonly referred to as the 'paradoxes of material implication' is essentially that when conditionals are analyzed in terms of material implication, a conditional will be true as long as the antecedent is false or the consequent is true. This produces some very counterintuitive results as there needs to be no relevant relation between the content of the antecedent and consequent, e.g.

(4) If it's 3:10pm, then Goldbach's conjecture is false.
(5) If everyone got 100 on the midterm exam, then there is no sugar in Diet Coke.
(6) If there are exactly two people are in this room, then there are exactly 100 people.

- Ideally, an analysis of natural language conditionals should avoid licensing any of these inferences while still validating various other inferences that *are* intuitively valid, e.g. modus ponens and modus tollens.

$$\frac{\phi \rightarrow \psi \qquad \phi}{\psi}\ \text{MP} \qquad\qquad \frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}\ \text{MT}$$

- Some people have alleged that there are counterexamples to MP and MT, but they are generally considered valid for natural language conditionals:

One famous alleged counterexample to *MP* is McGee (1985) and a more recent alleged counterexample to *MT* is Yalcin (2012).

(7) If Louise passed the exam, then John passed too. Louise passed. *Therefore*, John passed.
(8) If Louise passed the exam, then John passed too. John failed. *Therefore*, Louise failed.

- Conditional logics were initially conceived as an attempt to avoid these (or at least some) of these problems.

Strict Implication

- C. I. Lewis proposed that natural language conditionals be interpreted by use of a new connective, $\prec$, which Lewis defined as follows:

$$\phi \prec \psi =_{def} \neg\Diamond(\phi \wedge \neg\psi)$$

- On this analysis, a natural language conditional is equivalent to a necessary material implication, viz. $(\phi \prec \phi) \equiv \Box(\phi \rightarrow \psi)$. For this reason, this type of analysis is standardly referred to as the *strict implication* analysis.

  Proof. Assume $\neg\Diamond(\phi \wedge \neg\psi)$. Since $\Diamond$ is the dual of $\Box$, we get $\Box\neg(\phi \wedge \neg\psi)$. By DeMorgan, $\Box\neg(\phi \wedge \neg\psi)$ is equivalent to $\Box(\neg\phi \vee \neg\neg\psi)$ which is equivalent to $\Box(\neg\phi \vee \psi)$. So, by the definition of material implication, we get $\Box(\phi \rightarrow \psi)$.

- Instead of considering the merits of Lewis' axiomatic systems S1-S5, we will simply consider the merits of a strict implication analysis within normal modal logics (the problems that arise are essentially the same).

Avoiding the Paradoxes of Material Implication

- The main advantage of the strict implication analysis is that it helps resolve the paradoxes of material implication. Consider the strict implication versions of the paradoxes of material implication:

  A1. $\phi \prec (\psi \prec \phi)$

  A2. $\neg\phi \prec (\phi \prec \psi)$

- (A1) is equivalent to $\Box(\phi \rightarrow \Box(\psi \rightarrow \phi))$ and (A2) is equivalent to $\Box(\neg\phi \rightarrow \Box(\phi \rightarrow \psi))$. Both of these are invalid in all of the normal modal logics.

EXERCISE: Give countermodel in the weakest system possible.

- So, the strict implication analysis of conditionals avoids the predictions in (4)–(6).

Avoiding Negated Conditional to Conjunction

- The strict implication analysis also avoids the problem with inferences from negated conditionals to conjunctions. (A3) below is the strict implication version of that inference.

  A3. $\neg(\phi \prec \psi) \prec (\phi \wedge \neg\psi)$

- However, (A3) is equivalent to $\Box(\neg\Box(\phi \rightarrow \psi) \rightarrow (\phi \wedge \neg\psi))$ which is invalid in all the normal modal logics.

EXERCISE: Give countermodel in the weakest system possible.

Antecedent Strengthening and Contraposition

- As regards antecedent strengthening and contraposition, the strict implication analysis is unfortunately of no help as the formulas in A4. and A5. are both valid in every normal modal logic:

  A4. $(\phi \prec \psi) \prec ((\phi \wedge \chi) \prec \psi)$

  A5. $(\phi \prec \psi) \prec (\neg\psi \prec \neg\phi)$

EXERCISE: Give countermodel in the weakest system possible.

- This means that the strict implication analysis of conditionals is still going to make the predictions in (2) and (3).

The Paradoxes of Material Implication Redux

- Moreover, the paradoxes of material implication do not disappear *entirely* on the strict implication analysis. The strict implication analysis still validates formulas of the following kind.

Here $\bot$ is a necessary falsehood and $\top$ is a necessary truth

$$\bot \rightarrow \phi$$
$$\phi \rightarrow \top$$

- In other words, on the strict implication analysis, we predict conditionals with necessarily false antecedents or necessarily true consequents are always true, e.g.

(9) If every even number greater than 2 is not the sum of two primes, then no one would be surprised.
(10) If every even number greater than 2 is not the sum of two primes, then every even number greater than 2 is the sum of two primes.

Conditional Logic: The C Logics

- The conditional logic **C** is essentially an attempt at a formalization of an analysis of conditionals proposed by Stalnaker (1968).
- The general idea here is that a natural language conditional expresses a special type of necessity claim. So, we are going to need not only a modal semantics (that is, a model theory that includes possible worlds and accessibility relations), but also a new connective to represent natural language conditionals.
- So, we introduce a new binary connective, '>', governed by the following syntactic rule.

If ϕ and ψ are formulas, then so is $(\phi > \psi)$.

- The conceptual idea behind Stalnaker's analysis is that for a natural language conditional, if p then q, to be true, one needs only consider what is the case in possible worlds where p is true. If q is true in those worlds, the conditional is true.

In contrast to the *strict implication* analysis which says that if the conditional is true, then in every world either p is false or q is true.

- So, one way to implement this idea formally is to think of '$\phi > \psi$' as expressing a restricted necessity claim, viz. in every ϕ-world, ψ is true. We can express this idea in conditional logics by using a variation on the interpretation of $\Box$ and $\Diamond$ where these are *indexed* to a formula (viz. the antecedent of the conditional):

(11) $(\phi > \psi) \quad \rightsquigarrow \quad \Box_\phi \psi$

- That is, the $\Box$-formula in (11) is to be interpreted as the statement that ψ holds in every accessible ϕ-world. In other words, it is a necessity claim, but about a restricted set of possible worlds.

- A similar variation of $\Diamond$ can then be defined in terms of the $\Box$ in (11).

(12) $\Diamond_{\phi}\psi =_{def} \neg\Box_{\phi}\neg\psi$

- Hence, the formula '$\phi > \psi$' is to be read: *In all accessible ϕ-worlds, ψ is true.*

Again, this might seem very similar to the strict implication analysis, but there is one key difference, namely that on the strict analysis, accessible worlds where the antecedent is false are still considered relevant to the evaluation of the conditional. This is not the case given this alternative analysis.

- Since we are treating '$\phi > \psi$' as '$\Box_{\phi}\psi$', '$\neg(\phi > \psi)$' is going to be equivalent to $\neg\Box_{\phi}\psi$' which given the definition of $\Diamond$ above is equivalent to $\Diamond_{\phi}\neg\psi$.
- As is usual in modal logic, $\Diamond$ is a world-*generator* and $\Box$ as a world-*filler*.
- However, since in the current setup accessibility relations are *indexed* to formulas, we are going to use A to represent these (rather than the customary R).
- So, we can now state the truth conditions for '>' as follows:

(13) $(\phi > \psi)(w) = \Box_{\phi}\psi(w) = 1$ iff $\forall v(wA_{\phi}v \rightarrow \psi(v) = 1)$
(14) $\neg(\phi > \psi)(w) = \Diamond_{\phi}\neg\psi(w) = 1$ iff $\exists v(wA_{\phi}v \wedge \psi(v) = 0)$

- Using this semantics, we can now construct tree rules that will allow us to show the validity of various formulas in **C**.

Semantic Trees in C

- The tree rules for conditional logic **C** will consist of the standard rules for propositional logic (**PTr**) plus the two rules below.

It is imperative to note that in both these rules, the antecedent of the conditional and the subscript on the accessibility relation must match.

$$\begin{array}{lll} \phi > \psi & (w) & \checkmark \\ wA_{\phi}v & & \\ \vdots & & \\ \psi & (v) & (>\mathbf{R}) \end{array} \qquad \begin{array}{lll} \neg(\phi > \psi) & (w) & \checkmark \\ \vdots & & \\ wA_{\phi}v & & \\ \neg\psi & (v) & (\neg{>}\mathbf{R}) \end{array}$$

↑ where v is new to path

- So, the tree rules for conditional logic C are: **PTR** $\cup$ {>**R**, ¬>**R**}

- Let's consider a simple proof.
  - $(p > q), (p > r) \vDash_C p > (q \wedge r)$

1.	$p > q$	(w)	✓	**Pr**
2.	$p > r$	(w)	✓	**Pr**
3.	$\neg(p > (q \wedge r))$	(w)	✓	**NC**
4.	$wA_p v$			3, ¬>**R**
5.	$\neg(q \wedge r)$	(v)	✓	3, ¬>**R**
6.	q	(v)		1, 4, >**R**
7.	r	(v)		2, 4, >**R**
8.	$\neg q$ (v) × $\neg r$ (v) ×			5, **PL**

- In **C**, we can use semantic trees to demonstrate that *antecedent strengthening* (see Girle, p.97), *negated conditional to conjunction* (see Girle, p.96), and *contraposition* are all *invalid* (cf. below).
  - CONTRAPOSITION in **C**: $(p > q) > (\neg q > \neg p)$

1.	$\neg((p > q) > (\neg q > \neg p))$	(w)	✓	**NTF**
2.	$wA_{(p>q)}v$			1, ¬>**R**
3.	$\neg(\neg q > \neg p)$	(v)	✓	1,2, ¬>**R**
4.	$vA_{\neg q}u$			3, ¬>**R**
5.	$\neg\neg p$	(u)	✓	3,4, ¬>**R**
6.	p ↑	(u)		5, **PL**

- One countermodel would thus look as follows:
  - $W = \{w, v, u\}$
  - $A_{p>q} = \{\langle w,v \rangle\}$
  - $A_{\neg q} = \{\langle v,u \rangle\}$
  - I: $\{\langle\langle p,w\rangle,1\rangle, \langle\langle p,v\rangle,1\rangle, \langle\langle q,w\rangle,0\rangle, \langle\langle q,v\rangle,1\rangle, \langle\langle p,u\rangle,1\rangle, \langle\langle q,u\rangle,0\rangle\}$

Remember that we are analyzing contraposition as follows: $\Box_{\Box_p q}\Box_{\neg q}\neg p$

- So far so good. Conditional logic **C** ensures that these intuitively problematic inferences are not valid for >.
- However, the reason might just be that this logic is *too* weak. For example, in **C**, modus ponens fails for >.

- MODUS PONENS in **C**:

1.	$p > q$	(w)	**Premise**
2.	p	(w)	**Premise**
3.	$\neg q$	(v)	**NC**
	$\uparrow$		

- Given the rules are our disposal, there is no way forward in this tree. Hence, we are unable to close the tree. Predictably, we get the same result for modus tollens.
- While we do want to avoid *antecedent strengthening*, *contraposition*, *negated conditionals to conjunction*, and *the paradoxes of material implication* to hold for natural language conditionals, we do not an analysis that invalidates modus ponens and modus tollens.
- We thus need to amend **C** in order to avoid this result.

Conditional Logic $\mathbf{C}^+$

- The problem with **C** is similar to the problem with the normal modal logic *K*. Since there are no constraints on accessibility, access can only be generated by negated conditionals, and this is essentially the reason that MP and MT fail.
- So, in $\mathbf{C}^+$ we add a rule which allows us to evaluate the possible consequences of true and false conditionals at the world of evaluation, viz. a reflexivity rule.

$$(\phi > \psi) \quad (w)$$
$$\vdots$$
$$\neg\phi \quad (w) \qquad \Big| \qquad \phi \quad (w), \; wA_{\phi}w \qquad \textbf{Refl.}$$

(for any w in the path)

$$\neg(\phi > \psi) \quad (w)$$
$$\vdots$$
$$\neg\phi \quad (w) \qquad \Big| \qquad \phi \quad (w), \; wA_{\phi}w \qquad \textbf{Refl.}$$

(for any w in the path)

- In addition (Refl), we add another rule that, unlike the $\neg{>}\mathbf{R}$-rule, *makes explicit* that the antecedent of a false conditional, viz. $\neg(p > q)$, is true in the world that is made accessible, viz. the rule $\neg{>}\mathbf{R}^+$:

$$\begin{array}{lll} \neg(\phi > \psi) & (w) & \\ wA_{\phi}v & & \\ \phi & (v) & \\ \neg\psi & (v) & (\neg{>}\mathbf{R}^{+}) \end{array}$$

- With these rules added, it can now be shown that modus ponens is valid for $>$ in $\mathbf{C}^{+}$:

The same can be shown for modus tollens in $\mathbf{C}^{+}$, cf. Girle p.98-99.

1.	$p > q$	(w)	✓	**Premise**
2.	p	(w)		**Premise**
3.	$\neg q$	(w)		**NC**
4.	$\neg p$ (w) | p (w)			1, **Refl**
	× | $wA_p w$			
5.	| q (w)			1, 4, $>$**R**
	| ×			

- But as desired contraposition still fails for $>$ in $\mathbf{C}^{+}$.

1.	$\neg((p > q) > (\neg q > \neg p))$	(w)	✓	**NTF**
2.	$wA_{p>q}v$			1, $\neg{>}\mathbf{R}^{+}$
3.	$p > q$	(v)	✓	1, 2, $\neg{>}\mathbf{R}^{+}$
4.	$\neg(\neg q > \neg p)$	(v)	✓	1, 2, $\neg{>}\mathbf{R}^{+}$
5.	$vA_{\neg q}u$			4, $\neg{>}\mathbf{R}^{+}$
6.	$\neg q$	(u)		4, 5, $\neg{>}\mathbf{R}^{+}$
7.	$\neg p$	(u)		4, 5, $\neg{>}\mathbf{R}^{+}$
8.	$\neg p$ (v) | p (v)			**3, Refl**
	↑ | $vA_p v$			
9.	| q (v)			3, 8, $>$**R**
	| ↑			

- It can also be shown that *antecedent strengthening*, *negated conditional to conjunction*, and the *paradoxes of material implication* remain invalid in $\mathbf{C}^{+}$.

EXERCISES:

Check whether the following formulas are valid in C^+ using a semantic tree. If invalid, give a countermodel:

1. $(p > q) > ((p \wedge r) > q)$
2. $p > (q > p)$
3. $\big((p > q) \wedge (q > r)\big) > (p > r)$
4. $p > (\neg p > q)$
5. $\big((p \vee q) \wedge \neg q\big) > p)$

Logic 2: Modal Logics – Week 9

Deontic Logic

- Deontic Logic is the logic of obligations and permissions. So, in deontic logic, $\Box$ is interpreted as follows:

Strictly speaking, we should probably include an agent-parameter, but we will ignore this complications here.

$\Box\phi$ $\rightsquigarrow$ 'It is obligatory to bring it about that ϕ'
'One must bring it about that ϕ'
'One ought to bring it about that ϕ'

I will use these interchangeably throughout these notes.

- As usual, $\Diamond$ is the dual of $\Box$, so the interpretation of $\Diamond$ is the following:

$\Diamond\phi$ $\rightsquigarrow$ 'It's not obligatory to bring it about that not-ϕ'.

- Or equivalently:

$\Diamond\phi$ $\rightsquigarrow$ 'It is permissible to bring it about that ϕ'.
'One may bring it about that ϕ'

- As usual, we assume that the truth of '$\Box\phi$' requires ϕ to obtain in every accessible world (i.e. that ϕ has been brought about in every accessible world) and that the truth of '$\Diamond\phi$' requires that there is an accessible world where ϕ obtains (i.e. that ϕ has been brought about in at least one accessible world).

Ideal Worlds and Non-Reflexivity

- It is standardly assumed that the worlds quantified over in deontic logic are "ideal worlds", i.e. worlds where the laws/rules/regulations are obeyed.
- So, for a sentence such as 'it is obligatory to bring it about that ϕ' to be true, what is required is that ϕ is true in all the accessible worlds where these are construed as the ideal worlds. Similarly, for 'it is permissible to bring it about that ϕ' to be true, what is required is simply that there is an accessible (ideal) world where ϕ is true (where ϕ is brought about).
- With respect to deontic logic, it is immediately clear that any of the reflexive modal logics (e.g. T, S4, B, and S5) are going to make a bad prediction. These modal logics all include the axiom '$\Box\phi \rightarrow \phi$' — which model theoretically corresponds to a reflexivity constraint on the accessibility relation.
- Including this axiom in a deontic logic would mean that the actual world is always accessible, and hence that the actual world is included among the ideal worlds. As a result, for ϕ to be obligatory, it would have to be true. Or, in other words, if ϕ is not true, then its not obligatory.

- So, the only reasonable starting point for a logic of obligations and permissions is a non-reflexive logic. In standard deontic logic (which I'll refer to as SDL), the system D is preferred. The modal logic D is simply K with the following axiom added.

$$\mathbf{D}: \Box\phi \rightarrow \Diamond\phi$$

- Model-theoretically, this corresponds to adding a *seriality* constraint to the accessibility relation. Specifically, the accessibility relation must satisfy the following condition:

$$\forall w \exists v (wRv)$$

- The addition of this axiom to K ensures that an obligation to bring it about that ϕ entails the permissibility of bringing it about that ϕ.
- Following standard conventions, I will use '**O**' (obligation) is for '$\Box$' and '**P**' (permission) for '$\Diamond$'.

The Starting Point of Standard Deontic Logic

- In the original formulations deontic logic, the following axioms were assumed:

(**A1**) $\mathbf{O}p \leftrightarrow \neg\mathbf{P}\neg p$

(**A2**) $\mathbf{P}p \vee \mathbf{P}\neg p$

(**A3**) $\mathbf{P}(p \vee q) \leftrightarrow (\mathbf{P}p \vee \mathbf{P}q)$

(**A4**$*$) $\neg\mathbf{P}(p \wedge \neg p)$

(**A5**) $(p \leftrightarrow q) \rightarrow (\mathbf{P}p \leftrightarrow \mathbf{P}q)$

* This principle was actually rejected in the original formulation of basic deontic logic by von Wright.

- Brief explanation of these principles:

(**A1**) This is simply the definition of **P** as the dual of **O**.

(**A2**) This is the *principle of permission*. It states that for any proposition p, either it is permissible to bring about that p or it is permissible to bring about that $\neg p$. This seems intuitively reasonable.

(**A3**) This is the *principle of deontic distribution*. It states that if it is permissible to bring about p or q, then it is permissible to bring about p or it is permissible to bring about q — and vice versa, i.e. if it is permissible to bring about p or it is permissible to bring about q, then it is permissible to bring about p or q.

Notice that $\neg\mathbf{P}(p \wedge \neg p)$ is equivalent to $\mathbf{O}(p \vee \neg p)$.

(**A4**) This is essentially a *necessitation* principle: This states that it is not permissible to bring about a contradiction — or equivalently it is obligatory to bring about tautologies. This is admittedly an odd obligation, but it is trivially satisfied.

(**A5**) This is essentially a substitution principle: This states that if two propositions are logically equivalent, then the claim that one is permitted is logically equivalent to the claim that the other is permitted.

Tree Rules for Standard Deontic Logic (SDL)

- The previous axioms are all included in the standard deontic logic system **DT**. The tree rules for **DT** are the following:

- $\neg\mathbf{P}\phi \quad (w) \quad \checkmark$
  $\vdots$
  $\mathbf{O}\neg\phi \quad (w) \qquad$ **(OPN)**

- $\neg\mathbf{O}\phi \quad (w) \quad \checkmark$
  $\vdots$
  $\mathbf{P}\neg\phi \quad (w) \qquad$ **(OPN)**

- $\mathbf{P}\phi \quad (w) \quad \checkmark$
  $\vdots$
  wAv
  $\phi \quad (v) \qquad$ **(PR)**
  $\uparrow$
  where v is new to path.

- $\mathbf{O}\phi \quad (w) \quad \checkmark$
  $\vdots$
  $\mathbf{P}\phi \quad (w) \qquad$ **(OD)**

- $\mathbf{O}\phi \quad (w) \quad \checkmark$
  wAv
  $\vdots$
  $\phi \quad (v) \qquad$ **(OR)**

- If the following rule is added we get the system K4.

- $\mathbf{O}\phi \quad (w) \quad \checkmark$
  wAv
  $\vdots$
  $\mathbf{O}\phi \quad (v) \qquad$ **(OOR)**

- This is the rule corresponding to (Trans) in standard normal modal logics.

EXERCISES:
Consider the formulas below. Determine whether these are valid or invalid. If valid, show this using a semantic tree. If invalid, give a countermodel:

1. $\mathbf{O}(p \rightarrow q) \rightarrow (\mathbf{O}p \rightarrow \mathbf{O}q)$
2. $\mathbf{OP}p \vee \mathbf{PO}\neg p$
3. $\mathbf{P}p \rightarrow (\mathbf{O}q \rightarrow \mathbf{O}p)$

Puzzles in Standard Deontic Modal Logic

Ross' Paradox

These problems (and others) are nicely explicated in some recent lecture notes by Will Starr, cf. http://williamstarr.net/teaching/04.09.pdf.

- One of the most discussed problems for deontic logics is Ross' Paradox. This is the observation that the following inference is valid in SDL:

This is easily proved using the rules above.

(1) a. $\mathbf{O}p \rightarrow \mathbf{O}(p \vee q)$
b. $\mathbf{P}p \rightarrow \mathbf{P}(p \vee q)$

However, this means that inferences such as the following should be valid:

(2) I ought to clean my room.
So, I ought to clean my room or burn the house down.

(3) I am permitted to write an essay.
So, I am permitted to write an essay or plagiarize an essay.

- The extent to which these are a problem has been the subject of much debate. Girle seems to think that these are not much of a problem simply because the worlds quantified over in SDL are 'deontically perfect' worlds — and hence if there is an obligation to bring it about that p, then p is true in every world, and so $(p \vee q)$ is true in every world as well.

Material Implication and Conditional Obligations

- If conditionals are analyzed simply in terms of material implication, a number of very counterintuitive consequences follow in deontic logic. For example, in deontic logic, we get a variant of the paradox of material implication, namely the following:

(4) $\mathbf{O}\neg p \rightarrow \mathbf{O}(p \rightarrow q)$

- In other words, if p is forbidden (one is obligated to bring about not-p), then if one were to bring about p, one would be obligated to bring about anything and everything. But that seems absurd.

Free Choice Permission

- A problem that arises with respect to *permission* is that these seem to intuitively license something resembling the rule of simplification but for disjunction. Normally, from a disjunction, one may not infer either disjunct, i.e. $(p \vee q)$ does not entail p.
- But now consider (5). This sentence intuitively entails both (5a) and (5b).

(5) You may have ice cream or cake. (you are permitted ...)

a. You may have ice cream.
b. You may have cake.

- From (5a) and (5b), however, it seems we may infer (6).

(6) You may have ice cream and you may have cake.

- But this seems to suggest that you may have *both* ice cream and cake and it is not clear that this follows intuitively from (5).
- The problem is that it is not clear how to capture these facts. On the one hand, it looks as if we want our logic to license the following inferences:

$$\frac{\mathbf{P}(p \vee q)}{\mathbf{P}p} \qquad \frac{\mathbf{P}(p \vee q)}{\mathbf{P}q}$$

- Yet these inferences are clearly not going to be valid with our current set of rules.
- But even if these inferences could somehow be made to come out valid, we would then need some way of blocking the following inference:

$$\frac{\mathbf{P}(p \vee q)}{\mathbf{P}p \wedge \mathbf{P}q}$$

- And this would follow from the simple rule of conjunction introduction (conj.).

Forrester's Paradox

- Consider the statements below:

(7) It is forbidden for Jack to hit his son.
(8) If Jack does hit his son, he is obligated to hit him gently.
(9) Jack hits his son.

- Intuitively, there is nothing inconsistent about these sentences, i.e. it seems perfectly possible for these sentences to be simultaneously true.
- But now consider their translations into SDL:

Assume:
p = 'Jack hits his son'
q = 'Jack hits his son gently'

(10) $\mathbf{O}\neg p$
(11) $p \rightarrow \mathbf{O}q$
(12) p

- The problem is that these sentences are jointly inconsistent:
  - **Proof.** From (11) and (12) plus modus ponens, we get $\mathbf{O}q$. However, $q \vDash p$, viz. if Jack hit his son gently, it seems to follow that Jack hit his son. So, then from $\mathbf{O}q$ it follows that $\mathbf{O}p$ — and this is, of course, inconsistent with (10).
- The problem here is the substitution of logical entailments under **O**. However, in order to avoid this consequence, a quite radical departure from SDL is needed.

Chisholm's Paradox

- Consider the following set of intuitively consistent sentences:

Assume:
p = 'Jones goes'
q = 'Jones tells his neighbors he is coming'

(13) Jones ought to go (assist his neighbors).
(14) It ought to be that if Jones goes, he tells them he is coming.
(15) If Jones does not go, he ought to not tell them he is coming.
(16) As a matter of fact, Jones did not go

- Translating these into standard deontic logic, we get:

(17) $\mathbf{O}p$
(18) $\mathbf{O}(p \rightarrow q)$
(19) $\neg p \rightarrow \mathbf{O}\neg q$
(20) $\neg p$

- As in the case of Forrester's paradox (cf. above), we find that these sentences are inconsistent in standard deontic logic:
  - **Proof.** From (19) and (20) plus modus ponens, we get $\mathbf{O}\neg q$. From the proof we did in the exercise above, we know that $(\mathbf{O}p \rightarrow \mathbf{O}q)$ follows from (18). But this combined with (17) and modus ponens yields $\mathbf{O}q$ which is inconsistent with $\mathbf{O}\neg q$.

Must vs. Ought

- We have, so far, been treating both 'must' and 'ought' as having the same meaning, namely as **O**, viz.

$\mathbf{O}\phi$ $\rightsquigarrow$ 'It is obligatory to bring it about that ϕ'
'One must bring it about that ϕ'
'One ought to bring it about that ϕ'

- However, there are good reasons to suspect that 'must' and 'ought' have different meanings. For example, the sentence below seems clearly consistent.

(21) You may skip the talk, but you ought to attend.

- If we translate (21) into SDL, we get the following:

Assume:
p = 'you attend the talk'

(22) $\mathbf{P}\neg p \wedge \mathbf{O}p$

- But this is inconsistent.
  - **Proof.** From $\mathbf{P}\neg p$, it follows that there is an accessible $\neg p$-world but from $\mathbf{O}p$ it follows that every accessible world is a p-world. Contradiction.
- Also, notice that the sentence sounds weird with 'ought' substituted for 'must'.

(23) # You may skip the talk, but you must attend.

- So, in conclusion, it seems that we cannot treat 'ought' and 'must' as having the same quantificational force, viz. the same meaning.

The Miners Puzzle

- Recently a new puzzle about 'if' and 'ought' and the interaction of these were presented by Kolodny and MacFarlane (2010):

> "Ten miners are trapped either in shaft A or in shaft B, but we do not know which. Flood waters threaten to flood the shafts. We have enough sandbags to block one shaft, but not both. If we block one shaft, all the water will go into the other shaft, killing any miners inside it. If we block neither shaft, both shafts will fill halfway with water, and just one miner, the lowest in the shaft, will be killed." (K&M, 2010, p.1-2)

- Given this scenario, the following claims are all intuitively true.

(24) The miners are in shaft A or shaft B.

(25) If the miners are in shaft A, we ought to block A.

(26) If the miners are in shaft B, we ought to block B.

(27) We ought to block neither shaft.

> EXERCISE:
> Translate each of these sentences into formulas of standard deontic logic and show that they jointly lead to a contradiction.

Logic 2: Modal Logics – Revision Week

Modal Logics: Tree Rules, Axioms, and Countermodels

- Truth conditions for $\Box$ and $\Diamond$:
  - $\mathcal{V}^{\mathfrak{M}}(\Box\phi, w) = 1$ iff $\forall v(wRv \rightarrow \mathcal{V}^{\mathfrak{M}}(\phi, v) = 1)$
  - $\mathcal{V}^{\mathfrak{M}}(\Diamond\phi, w) = 1$ iff $\exists v(wRv \wedge \mathcal{V}^{\mathfrak{M}}(\phi, v) = 1)$

Basic Tree Rules for $\Box$ and $\Diamond$

$\Diamond$**R**-rule

$\Diamond\phi$	(w)	✓	
$\vdots$			
wRv			
ϕ	(v)		l.1, ($\Diamond$**R**)
	↑ *new world to path*		

NB! For the ($\Diamond$**R**) rule, it is still crucial that the world v introduced in line n is *new* to that part of the tree.

$\Box$**R**-rule

$\Box\phi$	(w)	✓	
wRv			
$\vdots$			
ϕ	(v)		l.1-2, ($\Box$**R**)

NB! For the ($\Box$**R**) rule, notice that antecedent access to some world is *required* in order to "discharge" the box.

Modal Negation Rules

- Since $\Box$ and $\Diamond$ are duals, we have the following rules for the interaction of negation with $\Box$ and $\Diamond$:

1.	$\neg\Diamond\phi$	(w)	✓	
$\vdots$	$\vdots$			
n	$\Box\neg\phi$	(w)		l.1, (**MN**)

1.	$\neg\Box\phi$	(w)	✓	
$\vdots$	$\vdots$			
n	$\Diamond\neg\phi$	(w)		l.1, (**MN**)

- The basic rules and the MN rules combined with the standard rules for propositional logic (**PTr**) yield the modal logic **K**.

Modal Logic T

- The modal logic **T** is characterized by having a reflexive accessibility relation, so we add the following tree rule:

Refl.

$\vdots$	
wRw	(Refl.)
↑ for any w in the path	

- In axiomatic approaches to modal logic, the axiom that characterizes **T** is the following:

$$\textbf{T}: \quad \Box\phi \rightarrow \phi$$

Modal Logic S4

- The modal logic **S4** is characterized by having a reflexive and transitive accessibility relation, so in addition to **(Refl.)** we add the following rule:

Trans.	wRv	
	vRu	
	$\vdots$	
	wRu	l.1, l.2, (Trans.)

- In axiomatic approaches to modal logic, the axiom that characterizes **S4** is the following:

$$\textbf{S4}: \quad \Box\phi \rightarrow \Box\Box\phi$$

Modal Logic B

- The modal logic **B** is characterized by having a reflexive and symmetric accessibility relation, so in addition to **(Refl.)** we add the following rule:

Symm.	wRv	
	$\vdots$	
	vRw	l.1, (Symm.)

- In axiomatic approaches to modal logic, the axiom that characterizes **B** is the following:

$$\textbf{B}: \quad \phi \rightarrow \Box\Diamond\phi$$

Modal Logic S5

- The modal logic **B** is characterized by having a reflexive, transitive, and symmetric accessibility relation, so the tree rules for S5 simply include $\Diamond$**R**, $\Box$**R**, **(Refl.)**, **(Trans.)**, and **(Symm.)**.

- In axiomatic approaches to modal logic, the axiom that characterizes **S5** is the following:

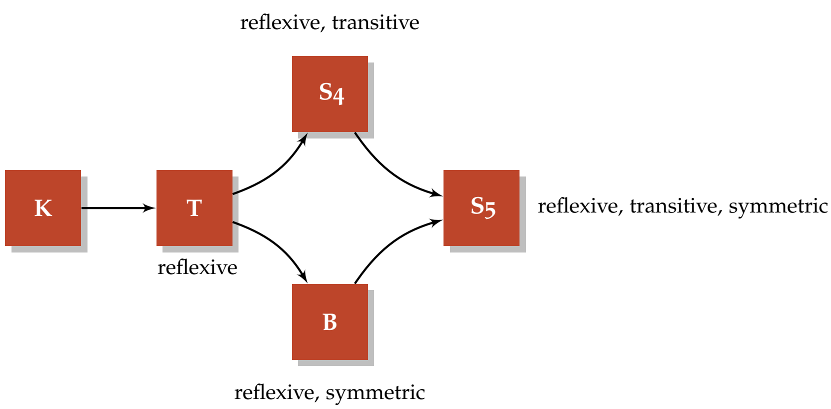
$$\textbf{S5}: \quad \Diamond\phi \rightarrow \Box\Diamond\phi$$

Summary of Tree Rules for the Normal Modal Logics

- The tree rules for each of the normal modal systems thus look as follows:

KTr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**}
TTr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**, **Refl.**}
S4Tr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**, **Refl.**, **Trans.**}
BTr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**, **Refl.**, **Symm.**}
S5Tr	=	**PTR** ∪ **MN** ∪ {◇**R**, □**R**, **Refl.**, **Trans.**, **Symm.**}

- Since **K** is the weakest of the systems, anything valid in **K** will be valid in every other system. Similarly, since **S5** is the strongest of the systems, anything invalid in **S5** will be invalid in every weaker systems. The diagram below indicates the relative strength of the normal modal logics:



Epistemic Logic

- The tree rules for epistemic logic are more or less identical to the rules in standard modal logic — the main difference being that the modal operators, K and P, and the accessibility relation, E, are relativized to agents.

- Truth Conditions for K and P:

  - $\mathcal{V}^{\mathfrak{M}}(K_x\phi, w) = 1$ iff $\forall v(wE_xv \rightarrow \mathcal{V}^{\mathfrak{M}}(\phi, v) = 1)$
  - $\mathcal{V}^{\mathfrak{M}}(P_x\phi, w) = 1$ iff $\exists v(wE_xv \wedge \mathcal{V}^{\mathfrak{M}}(\phi, v) = 1)$

Tree Rules for Epistemic Logic

KPN-rules	$\neg K_x\phi$	(w)	✓	$\neg P_x\phi$	(w)	✓	
	⋮			⋮			
	$P_x\neg\phi$	(w)	(KPN)	$K_x\neg\phi$	(w)		(KPN)

PR $P_x\phi$ (w) ✓
⋮
wE_xv
ϕ (v) (PR)
↑
where v is new to path.

KR $K_x\phi$ (w) ✓
wE_xv
⋮
ϕ (v) (KR)
↑
for any v.

KT $K_x\phi$ (w) ✓
⋮
ϕ (w) (KT)

KKR $K_x\phi$ (w) ✓
wE_xv
⋮
$K_x\phi$ (v) (KKR)
↑
for any v.

- The **KT**-rule is the correlate of *reflexivity* guaranteeing factivity, hence with this rule added we get an epistemic logic based on modal logic **T**:
  - Tree Rules for Epistemic Logic **T** = {**PTr, KPN**, **PR**, **KR**, **KT**}.
- The **KKR**-rule is the correlate of transitivity guaranteeing the so-called KK-principle, viz. $K_x\phi \to K_xK_x\phi$. Thus, if this rule is added to the tree rules for **T**, we get the epistemic logic **S4**.
  - Tree Rules for Epistemic Logic **S4** = {**PTr, KPN**, **PR**, **KR**, **KT**, **KKR**}.
- For multiagent epistemic logic, we might add the following tree rule as well:

TrKR $K_xK_y\phi$ (w) ✓
⋮
$K_x\phi$ (w) (TrKR)

Deontic Logic

- Similar to Epistemic Logic, we are only going to consider the tree rules for two deontic modal logics, namely **DT** and **D4**.
- In deontic logic, as usual we have two modal operators **O** (obligation) and **P** (permission) which have the standard truth conditions:
  - $\mathcal{V}^{\mathfrak{M}}(\mathbf{O}\phi, w) = 1$ iff $\forall v(wAv \to \mathcal{V}^{\mathfrak{M}}(\phi, v) = 1)$
  - $\mathcal{V}^{\mathfrak{M}}(\mathbf{P}\phi, w) = 1$ iff $\exists v(wAv \wedge \mathcal{V}^{\mathfrak{M}}(\phi, v) = 1)$

Tree Rules for Deontic Logic

OPN $\neg\mathbf{P}\phi$ (w) √
⋮
$\mathbf{O}\neg\phi$ (w) (OPN)

OPN $\neg\mathbf{O}\phi$ (w) √
⋮
$\mathbf{P}\neg\phi$ (w) (OPN)

PR $\mathbf{P}\phi$ (w) √
⋮
wAv
ϕ (v) (PR)
↑
where v is new to path.

OR $\mathbf{O}\phi$ (w) √
wAv
⋮
ϕ (v) (OR)

OD $\mathbf{O}\phi$ (w) √
⋮
$\mathbf{P}\phi$ (w) (OD)

OOR $\mathbf{O}\phi$ (w) √
wAv
⋮
$\mathbf{O}\phi$ (v) (OOR)

- The key thing to remember about the deontic logics is that these are non-reflexive logics. So, the name **DT** is slightly misleading as none of the tree rules above correspond to a *reflexivity* constraint on the accessibility relation.
- Although deontic logics are non-reflexive, they do include the rule **OD**: This corresponds to a *seriality* constraint on the accessibility relation. The axiom that characterizes modal logics with a serial accessibility relation is the following:

$$\Box\phi \rightarrow \Diamond\phi$$

  - Tree Rules for **DT**: {**PTr**, **OPN**, **PR**, **OR**, **OD**}.
- The rule **OOR** is the correlate of transitivity in deontic logic. This rule validates the following formula: $\mathbf{O}\phi \rightarrow \mathbf{OO}\phi$.
- When the **OOR** rule is added to **DT**, we get the system **D4**.
  - Tree Rules for **D4**: {**PTr**, **OPN**, **PR**, **OR**, **OD**, **OOR**}.

Conditional Logic C^+

- Conditional Logic C^+ is a type of modal propositional logic. However, instead of $\Box$ and $\Diamond$, C^+ has just one (binary) modal operator, namely $>$.

- The semantics for $>$ is the following:
  - $\mathcal{V}^{\mathfrak{M}}(\phi > \psi, w) = 1$ iff $\forall v(wA_{\phi}v \rightarrow \mathcal{V}^{\mathfrak{M}}(\psi, v) = 1)$
  - $\mathcal{V}^{\mathfrak{M}}(\neg(\phi > \psi), w) = 1$ iff $\exists v(wA_{\phi}v \wedge \mathcal{V}^{\mathfrak{M}}(\psi, v) = 0)$

The notation '$wA_{\phi}v$' here means that v is an accessible ϕ-world from w.

Semantic Trees in Conditional Logic C^{+}

- In addition the standard tree rules for propositional logic (PTr), we add the following rules in C^{+}:

>R

$$\begin{array}{lll} \phi > \psi & (w) & \checkmark \\ wA_{\phi}v & & \\ \vdots & & \\ \psi & (v) & (>\text{R}) \end{array}$$

¬>R

$$\begin{array}{lll} \neg(\phi > \psi) & (w) & \checkmark \\ \vdots & & \\ wA_{\phi}v & & \\ \neg\psi & (v) & (\neg>\text{R}) \end{array}$$

↑ where v is new to path

Refl.

$$\begin{array}{c} (\phi > \psi) \quad (w) \\ \vdots \\ \swarrow \qquad \searrow \\ \begin{array}{ll} \neg\phi \ (w) & \begin{array}{c} \phi \ (w) \\ wA_{\phi}w \end{array} \end{array} \quad (\text{Refl.}) \end{array}$$

(for any w in the path)

Refl.

$$\begin{array}{c} \neg(\phi > \psi) \quad (w) \\ \vdots \\ \swarrow \qquad \searrow \\ \begin{array}{ll} \neg\phi \ (w) & \begin{array}{c} \phi \ (w) \\ wA_{\phi}w \end{array} \end{array} \quad (\text{Refl.}) \end{array}$$

(for any w in the path)

¬>R^{+}

$$\begin{array}{lll} \neg(\phi > \psi) & (w) & \\ wA_{\phi}v & & \\ \phi & (v) & \\ \neg\psi & (v) & (\neg>\mathbf{R}^{+}) \end{array}$$

- Tree Rules for Conditional Logic C^{+}: {**PTr**, >**R**, ¬>**R**, ¬>**R**$^{+}$, **Refl.**}

Countermodels

- To show that a formula is invalid one constructs a model in which the target formula is false. Since validity consists in truth in every world of every model (for the relevant system), the existence of a such a model (viz. a countermodel) is proof that the formula is invalid.

Countermodels in MPL

- A model in modal propositional logic consists of the following:
  - A non-empty set of worlds.
  - A (possibly empty) set of accessibility relations.
  - A specification of the interpretation function, i.e. an assignment of truth values to the sentence letters in the formula relative to each world of the model.
- The modal system in question will constrain which accessibility relations must be included.

Countermodels in QML

- In quantified modal logic, models are more involved than in propositional modal logic. A model must include:
  - A domain of individuals.
  - A non-empty set of worlds (possibly a singleton set).
  - A set of accessibility relations (possibly empty).
  - An interpretation function which specifies:
    a. The extension of the relevant constants: a, b, c, ...
    b. The extension of the relevant predicates relative to the relevant worlds: F^1, F^2, F^3, ...
    c. The extension of the relevant relations relative to the relevant worlds: R^1, R^2, R^3, ...
  - Again, the modal system in question will constrain which accessibility relations must be included.

If the modal logic in question has variable domains, then world-specific domains need to be specified.

Countermodels in Propositional Epistemic Logic

- Countermodels in epistemic propositional logic are very similar to countermodels in standard modal propositional logic except that the accessibility relations must be agent-relative. So, a model in epistemic logic must include:
  - A non-empty set of worlds (possibly a singleton set).
  - A (possibly empty) set of accessibility relations relativized to the relevant agent(s).
  - A specification of the interpretation function, i.e. an assignment of truth values to the sentence letters in the formula relative to each world of the model.

Countermodels in Propositional Deontic Logic

- Countermodels in deontic propositional logic are structurally identical to countermodels in standard modal propositional logic, i.e. these must include a set of worlds, set of accessibility relations, and an interpretation function.

Countermodels in Conditional Logic C^+

- Countermodels in Conditional Logic C^+ must contain the following:
  - A non-empty set of possible worlds.
  - A non-empty set of accessibility relations.
  - An interpretation function assigning truth values to each sentence letter in the relevant formula relative to each possible world in the model.

Notice that the set of accessibility relations cannot be empty in deontic logic as there is a *seriality* constraint on the frames.

Logic 2: Modal Logics — Exercise Booklet

Translations: Exercises

- Translate the following sentences into formulas of *Modal Propositional Logic*.

1. It is possible that it might rain.
2. If Sue runs for office, Louise might run too.
3. We must block door 1 or door 2.
4. To start the engine, the key must be turned.
5. The garbage truck can only lift the bins if they are closed.
6. Sue must not be happy.
7. If parents routinely question their doctor, they might not do what is right for their child.
8. Fred or Mary might have stolen the diamonds, but not both.

Solutions on the next pages...

Translations: Solutions

1. It is possible that it might rain.

   (a) $\Diamond\Diamond p$

   (b) $\Diamond p$

p: 'it rains'.

One could argue that the second modal is superfluous in English, i.e. it doesn't actually change the meaning and hence translate this with just one $\Diamond$. That would be acceptable in this case.

2. If Sue runs for office, Louise might run too.

   (a) $p \to \Diamond q$

p: 'Sue runs for office'
q: 'Louise runs for office'

· Follow up question: Why is this translation better than '$\Diamond(p \to q)$'?

3. We must block door 1 or door 2.

   (a) $\Box(p \vee q)$

   (b) $\Box p \vee \Box q$

p: 'We block door 1'
q: 'We block door 2'

Both these translations are correct, the sentence is ambiguous. Notice that these do not mean the same thing. In (a), we have an obligation, but are free to choose how to satisfy this obligation. In (b), we also have an obligation, but the sentence could be true even if we only have one specific obligation, e.g. to see to it that p.

4. To start the engine, the key must be turned.

   (a) $\Box(p \to q)$

p: 'The engine starts'
q: 'The key is turned'

Notice that this conditional states a *necessary* condition, not a *sufficient* condition. In other words, it is necessary for starting the engine that the key is turned. Hence, necessarily, if the engine started, the key was turned.

· Follow up question: Why is this translation better than '$p \to \Box q$'?

5. The garbage truck can only lift the bins if they are closed.

   (a) $\Diamond p \to q$

p: 'The garbage truck lifts the bins'
q: 'The bins are closed'

Again, this conditional states a necessary condition, namely that in order for it to be possible for the bins to be lifted by the truck, the have to be closed.
It follows that if it is possible for the trucks to lift the bins, they are closed.

6. Sue must not be happy.

   (a) $\Box\neg p$

p: 'Sue is happy'

7. If parents routinely question their doctor, they might not do what is right for their child.

   (a) $p \rightarrow \Diamond\neg q$

p: 'parents routinely question their doctor'
q: 'parents do what is right for their child'

8. Fred or Mary might have stolen the diamonds, but not both.

   (a) $(\Diamond p \wedge \Diamond q) \wedge \neg\Diamond(p \wedge q)$

p: 'Fred stole the diamonds'
q: 'Mary stole the diamonds'

Semantic Trees: Exercises

- Using semantic trees, show the following formulas are semantically valid. For these you may use the **PL**-rules, **MN**-rules, □**S5**-rule, and ◇**S5**-rule.

  1. $\vDash_{S5} (\Diamond p \to \Box\Diamond p)$
  2. $\vDash_{S5} \Box(p \to q) \to \Box(\Box p \to \Box q)$
  3. $\vDash_{S5} (\Diamond\Diamond p \to \Diamond p)$
  4. $\vDash_{S5} \Diamond(p \vee q) \leftrightarrow \neg(\neg\Diamond p \wedge \neg\Diamond q)$
  5. $\vDash_{S5} (\Box p \to p)$
  6. $\vDash_{S5} \Box p \to \neg\Diamond\neg\Box p$
  7. $\vDash_{S5} \Box\Box(p \to p)$
  8. $\vDash_{S5} (\Box(q \to p) \wedge \Box(\neg q \to p)) \leftrightarrow \Box p$

- For these, you may only use the **PL**-rules, **MN**-rules, □**R**-rule, ◇**R**-rule, and also the rules (**Refl.**), (**Trans.**), and (**Symm.**) when appropriate.

  That is, you are not allowed to use the □**S5** and ◇**S5**.

  9. $\vDash_{K} \Box\neg p \to \Box(p \to q)$
  10. $\vDash_{T} \Box\Box p \to \Diamond p$
  11. $\vDash_{T} \neg\Diamond(p \vee q) \to \neg\Diamond\Box q$
  12. $\vDash_{S4} \Diamond\Box\Box p \vee \neg\Box p$
  13. $\vDash_{S4} \Box p \to \Box\Diamond\Box p$

- **Additional Exercises**
  If you've finished the above exercises, go back and do 1-8 without using the □**S5**-rule or the ◇**S5**-rule, but □**R**-rule and (**Refl.**), (**Trans.**), (**Symm.**) instead.

Solutions on the next pages...

Semantic Trees: Solutions

Rules admissible for 1-8: **PL**, **MN**, $\Box$**S5**, $\Diamond$**S5**.

1. $\vDash_{S5} (\Diamond p \to \Box\Diamond p)$

1.	$\neg(\Diamond p \to \Box\Diamond p)$	(w)	NTF
2.	$\Diamond p$	(w)	l.1, PL
3.	$\neg\Box\Diamond p$	(w)	l.1, PL
4.	p	(v)	l.2, $\Diamond$**S5**
5.	$\Diamond\neg\Diamond p$	(w)	l.3, MN
6.	$\neg\Diamond p$	(u)	l.5, $\Diamond$**S5**
7.	$\Box\neg p$	(u)	l.6, MN
8.	$\neg p$	(v)	l.7, $\Box$**S5**
	$\times$		

2. $\vDash_{S5} \Box(p \to q) \to \Box(\Box p \to \Box q)$

1.	$\neg(\Box(p \to q) \to \Box(\Box p \to \Box q))$	(w)	NTF
2.	$\Box(p \to q)$	(w)	l.1, **PL**
3.	$\neg\Box(\Box p \to \Box q)$	(w)	l.1, **PL**
4.	$\Diamond\neg(\Box p \to \Box q)$	(w)	l.3, **MN**
5.	$\neg(\Box p \to \Box q)$	(v)	l.4, $\Diamond$**S5**
6.	$\Box p$	(v)	l.5, **PL**
7.	$\neg\Box q$	(v)	l.5, **PL**
8.	$\Diamond\neg q$	(v)	l.6, **MN**
9.	$\neg q$	(u)	l.8, $\Diamond$**S5**
10.	p	(u)	l.6, $\Box$**S5**
11.	$p \to q$	(u)	l.2, $\Box$**S5**
12.	$\neg p$ (u) $\times$ | q (u) $\times$		l.11, **PL**

3. $\vDash_{S5} (\Diamond\Diamond p \to \Diamond p)$

1.	$\neg(\Diamond\Diamond p \to \Diamond p)$	(w)	NTF
2.	$\Diamond\Diamond p$	(w)	1, **PL**
3.	$\neg\Diamond p$	(w)	1, **PL**
4.	$\Diamond p$	(v)	2, $\Diamond$**S5**
5.	p	(u)	4, $\Diamond$**S5**
6.	$\Box\neg p$	(w)	3, **MN**
7.	$\neg p$	(u)	6, $\Box$**S5**
	$\times$		

4. $\vDash_{S5} \Diamond(p \vee q) \leftrightarrow \neg(\neg\Diamond p \wedge \neg\Diamond q)$

1.	$\neg(\Diamond(p \vee q) \leftrightarrow \neg(\neg\Diamond p \wedge \neg\Diamond q))$ (w)		NTF
2.	$\Diamond(p \vee q)$ (w) $\neg\neg(\neg\Diamond p \wedge \neg\Diamond q)$ (w)	$\neg\Diamond(p \vee q)$ (w) $\neg(\neg\Diamond p \wedge \neg\Diamond q)$ (w)	1, **PL**
4.	$\neg\Diamond p \wedge \neg\Diamond q$ (w)		4, **PL**
5.	$\neg\Diamond p$ (w)		l.4, **PL**
6.	$\neg\Diamond q$ (w)		l.4, **PL**
7.	$\Box\neg p$ (w)		l.6, **PL**
8.	$\Box\neg q$ (w)		l.7, **PL**
9.	$p \vee q$ (v)		l.2, $\Diamond$**S5**
10.	p (v) / q (v)		9, **PL**
11.	$\neg p$ (v) / $\neg q$ (v) × / ×		l.7, l.8, $\Box$**S5**
12.			

left as exercise – check your answer at office hours

5. $\vDash_{S5} (\Box p \rightarrow p)$

1.	$\neg(\Box p \rightarrow p)$	(w)	NTF
2.	$\Box p$	(w)	l.1, **PL**
3.	$\neg p$	(w)	l.1, **PL**
4.	p	(w)	l.2, $\Box$**S5**
	×		

6. $\vDash_{S5} \Box p \rightarrow \neg\Diamond\neg\Box p$

1.	$\neg(\Box p \rightarrow \neg\Diamond\neg\Box p)$	(w)	NTF
2.	$\Box p$	(w)	l.1, **PL**
3.	$\neg\neg\Diamond\neg\Box p$	(w)	l.1, **PL**
4.	$\Diamond\neg\Box p$	(w)	l.3, **PL**
5.	$\neg\Box p$	(v)	l.4, $\Diamond$**S5**
6.	$\Diamond\neg p$	(v)	l.5, **MN**
7.	$\neg p$	(u)	l.6, $\Diamond$**S5**
8.	p	(u)	l.2, $\Box$**S5**
	×		

7. $\vDash_{S5} \Box\Box(p \to p)$

1.	$\neg\Box\Box(p \to p)$	(w)	NTF
2.	$\Diamond\neg\Box(p \to p)$	(w)	l.1, **MN**
3.	$\neg\Box(p \to p)$	(w)	l.2, $\Diamond$**S5**
4.	$\Diamond\neg(p \to p)$	(w)	l.3, **MN**
5.	$\neg(p \to p)$	(w)	l.4, $\Diamond$**S5**
6.	p	(w)	l.5, **PL**
7.	$\neg p$	(w)	l.5, **PL**
	×		

8. $\vDash_{S5} (\Box(q \to p) \wedge \Box(\neg qi \to p)) \to \Box p$

1.	$\neg((\Box(q \to p) \wedge \Box(\neg q \to p)) \to \Box p)$		(w)	NTF
2.	$(\Box(q \to p) \wedge \Box(\neg q \to p))$		(w)	l.1, **PL**
3.	$\neg\Box p$		(w)	l.1, **PL**
4.	$\Box(q \to p)$		(w)	l.2, **PL**
5.	$\Box(\neg q \to p)$		(w)	l.2, **PL**
6.	$\Diamond\neg p$		(w)	l.3, **MN**
7.	$\neg p$		(v)	l.6, $\Diamond$**S5**
8.	$q \to p$		(v)	l.4, $\Box$**S5**
9.	$\neg q \to p$		(v)	l.5, $\Box$**S5**
10.	$\neg q$	p ×	(v)	8, **PL**
11.	$\neg\neg q$ × p ×		(v)	l.9, **PL**

Rules admissible for 9-13: **PL**, **MN**, $\Box$**R**, $\Diamond$**R**. The rules (**Refl.**), (**Trans.**), and (**Symm.**) should also be used when appropriate.

9. $\vDash_{K} \Box\neg p \to \Box(p \to q)$

1.	$\neg(\Box\neg p \to \Box(p \to q))$	(w)	NTF
2.	$\Box\neg p$	(w)	l.1
3.	$\neg\Box(p \to q)$	(w)	l.1
4.	$\Diamond\neg(p \to q)$	(w)	l.3, MN
5.	wRv		l.4, $\Diamond$**R**
6.	$\neg(p \to q)$	(v)	4-5, $\Diamond$**R**
7.	p	(v)	l.6
8.	$\neg q$	(v)	l.6
9.	$\neg p$	(v)	l.2, l.5, $\Box$**R**
	×		

10. $\vDash_{T} \Box\Box p \to \Diamond p$

1.	$\neg(\Box\Box p \to \Diamond p)$	(w)	NTF
2.	$\Box\Box p$	(w)	l.1
3.	$\neg\Diamond p$	(w)	l.1
4.	$\Box\neg p$	(w)	l.3, MN
5.	wRw		(refl.)
6.	$\neg p$	(w)	l.4, l.5, $\Box$**R**
7.	$\Box p$	(w)	l.2,l.5
8.	p	(w)	l.5, l.7, $\Box$**R**
	$\times$		

11. $\vDash_{T}$ $\neg\Diamond(p \vee q) \to \neg\Diamond\Box q$

1.	$\neg(\neg\Diamond(p \vee q) \to \neg\Diamond\Box q)$	(w)	NTF
2.	$\neg\Diamond(p \vee q)$	(w)	l.1
3.	$\neg\neg\Diamond\Box q$	(w)	l.1
4.	$\Diamond\Box q$	(w)	l.3
5.	wRv		l.4, $\Diamond$**R**
6.	$\Box q$	(v)	l.4, l.5,$\Diamond$**R**
7.	$\Box\neg(p \vee q)$	(w)	l.2, MN
8.	$\neg(p \vee q)$	(v)	l.7, l.5, $\Box$**R**
9.	$\neg p$	(v)	l.8
10.	$\neg q$	(v)	l.8
11.	vRv		(Refl.)
12.	q	(v)	l.6, l.11, $\Box$**R**
	$\times$		

12. $\vDash_{S4}$ $\Diamond\Box\Box p \vee \neg\Box p$

1.	$\neg(\Diamond\Box\Box p \vee \neg\Box p)$	(w)	NTF
2.	$\neg\Diamond\Box\Box p$	(w)	l.1
3.	$\neg\neg\Box p$	(w)	l.1
4.	$\Box\neg\Box\Box p$	(w)	l.2, MN
5.	wRw	(w)	(Refl.)
6.	$\neg\Box\Box p$	(w)	l.4, l.5, $\Box$**R**
7.	$\Diamond\neg\Box p$	(w)	l.6, MN
8.	wRv		l.7, $\Diamond$**R**
9.	$\neg\Box p$	(v)	l.7, l.8, $\Diamond$**R**
10.	$\Diamond\neg p$	(v)	l.9, MN
11.	vRu		l.10, $\Diamond$**R**
12.	$\neg p$	(u)	l.10, l.11, $\Diamond$**R**
13.	$\Box p$	(w)	l.3
14.	wRu		l.8, l.11, (Trans.)
15.	p	(u)	l.13, l.14, $\Box$**R**
	$\times$		

13. $\vDash_{S4} \Box p \to \Box\Diamond\Box p$

1.	$\neg(\Box p \to \Box\Diamond\Box p)$	(w)	NTF
2.	$\Box p$	(w)	l.1
3.	$\neg\Box\Diamond\Box p$	(w)	l.1
4.	$\Diamond\neg\Diamond\Box p$	(w)	l.3, MN
5.	wRv		l.4, $\Diamond$**R**
6.	$\neg\Diamond\Box p$	(v)	l.4, l.5, $\Diamond$**R**
7.	$\Box\neg\Box p$	(v)	l.7, MN
8.	vRv		(Refl.)
9.	$\neg\Box p$	(v)	l.7, l.8, $\Box$**R**
10.	$\Diamond\neg p$	(v)	l.9, MN
11.	vRu		l.10, $\Diamond$**R**
12.	$\neg p$	(u)	l.10, l.11, $\Diamond$**R**
13.	wRu		l.5, l.11, (Trans.)
14.	p	(u)	l.2, l.13, $\Box$**R**
	$\times$		

Translations and Semantic Trees: Exercises

- Translate these arguments into *Modal Propositional Logic* and show that they are S5-valid using semantic trees.

Argument 1: God exists
God's existence is either necessary or impossible. Moreover, God's existence is conceivable. If God's existence is conceivable, then his existence is possible. So, God exists.

p: 'God exists'
q: 'God's existence is conceivable'

Argument 2: There is no external world.
If I have hands, there must be an external world. If there is an external world, then I cannot be a brain in a vat. It's possible that I'm a brain in a vat. So, it is possible that I do not have hands.

p: 'I have hands'
q: 'There is an external world'
r: 'I'm a brain in a vat'

Solutions on the next pages...

Argument 1, Translation: Solution

Premise 1:	$\Box p \vee \neg\Diamond p$
Premise 2:	q
Premise 3:	$q \to \Diamond p$
Conclusion:	p

Argument 1, Semantic Tree: Solution

1.	$\Box p \vee \neg\Diamond p$	(w)	Premise
2.	q	(w)	Premise
3.	$q \to \Diamond p$	(w)	Premise
4.	$\neg p$	(w)	NC
5.	$\neg q \quad (w)$ ×	$\Diamond p \quad (w)$		l.3, PL
6.	(under $\Diamond p$:) $\Box p \quad (w)$	$\neg\Diamond p \quad (w)$ ×		l.1, PL
7.	wRw		Refl.
8.	p ×		l.6-7, $\Box$**R**

Argument 2, Translation: Solution

Premise 1:	$\Box(p \to q)$
Premise 2:	$\Box(q \to \neg r)$
Premise 3:	$\Diamond r$
Conclusion:	$\Diamond\neg p$

Argument 2, Semantic Tree: Solution

1.	$\Box(p \to q)$	(w)	Premise
2.	$\Box(q \to \neg r)$	(w)	Premise
3.	$\Diamond r$	(w)	Premise
4.	$\neg\Diamond\neg p$	(w)	NC
5.	$\Box p$	(w)	l.4, MN
6.	wRv		l.3, $\Diamond$**R**
7.	r	(v)	l.3, $\Diamond$**R**
8.	p	(v)	l.5,l.6, $\Diamond$**R**
9.	$p \to q$	(v)	l.1,l.6, $\Box$**R**
10.	$q \to \neg r$	(v)	l.2,l.6, $\Box$**R**
11.	$\neg p \quad (v)$ ×	$q \quad (v)$		l.9, PL
12.	(under q:) $\neg q \quad (v)$ ×	$\neg r \quad (v)$ ×		l.10, PL

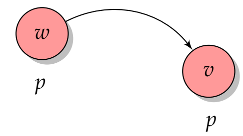
Countermodels: Exercises

1. $\nvDash_{K} \neg(\Box\neg p \to \Box(p \to \neg p))$
2. $\nvDash_{S4} \neg\big((\Diamond p \lor \Box q) \lor \neg\Diamond\Diamond p\big)$
3. $\nvDash_{S5} \neg\big(\neg\Box(\Box p \to \Box q) \to \neg\Box(p \to q)\big)$
4. $\nvDash_{S5} (\Diamond(p \lor q) \to \Diamond r) \to \Box(p \lor \Diamond q)$

Solutions on the next pages...

Countermodels: Solutions

1. $\nvDash_{K} \neg(\Box\neg p \to \Box(p \to \neg p))$



This is a negated conditional, so it is true only if the antecedent is true and the consequent false. So, to find a countermodel we simply need to find a model where either the antecedent is false or the consequent true.

MODEL	$\mathcal{W}$	w, v
	$\mathcal{R}$	wRv
	$\mathcal{I}$	$\langle\langle p,w\rangle,1\rangle$, $\langle\langle p,v\rangle,1\rangle$

Since in this model $\mathcal{V}_{\mathfrak{M}}(p,v) = 1$ and $wRv \in \mathcal{R}$, this means that $\mathcal{V}_{\mathfrak{M}}(\Box\neg p,w) = 0$. So, the antecedent of the conditional is false, which means that the conditional is true. This in turn means that the negation of the conditional is false. And hence we have a countermodel.

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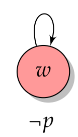
2. $\nvDash_{S4} \neg\big((\Diamond p \vee \Box q) \vee \neg\Diamond\Diamond p\big)$



Since this is a negated disjunction, by DeMorgan it is equivalent to $\neg(\Diamond p \vee \Box q) \wedge \neg\neg\Diamond\Diamond p$. So, to find a countermodel, we simply need to find an S4-model where one of these conjuncts are false.

MODEL	$\mathcal{W}$	w
	$\mathcal{R}$	wRw
	$\mathcal{I}$	$\langle\langle p,w\rangle,0\rangle$

In this model, $\mathcal{V}_{\mathfrak{M}}(\Diamond\Diamond p, w = 0)$, so it follows that $\mathcal{V}_{\mathfrak{M}}(\neg\neg\Diamond\Diamond p, w) = 0$. Given this, one of the conjuncts are false, so the conjunction as a whole is false. Hence we have a countermodel.

NB! This counts as an **S4-model** since the transitivity condition on $\mathcal{R}$ is satisfied. That condition is:

$$\forall x \forall y \forall z \big((xRy \wedge yRz) \to xRz\big)$$

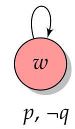
But since there is only one world w, there is only one possible instantiation:

$$(wRw \wedge wRw) \to wRw.$$

And this condition is clearly satisfied.

3. $\not\models_{S5} \neg\big(\neg\Box(\Box p \to \Box q) \to \neg\Box(p \to q)\big)$

This is a negated conditional, so it is true only if the antecedent is true and the consequent false. So, to find a countermodel we simply need to find a model where either the antecedent is false or the consequent true.



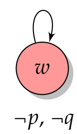
MODEL		
	$\mathcal{W}$	w
	$\mathcal{R}$	wRw
	$\mathcal{I}$	$\langle\langle p,w\rangle,1\rangle$, $\langle\langle q,w\rangle,0\rangle$

In this model, $\mathcal{V}_{\mathfrak{M}}((p \to q), w) = 0$. Consequently, $\mathcal{V}_{\mathfrak{M}}(\neg(p \to q), w) = 1$. Since wRw, it follows that $\mathcal{V}_{\mathfrak{M}}(\Diamond\neg(p \to q), w) = 1$ and this, of course is equivalent to $\mathcal{V}_{\mathfrak{M}}(\neg\Box(p \to q), w) = 1$. This means that the conditional is true, and hence that the negation of the conditional false. So, we have a countermodel.

---

4. $\not\models_{S5} (\Diamond(p \vee q) \to \Diamond r) \to \Box(p \vee \Diamond q)$

To find a countermodel here, we need a model where the antecedent of the conditional is true and the consequent false. Again, we only need one world.



MODEL		
	$\mathcal{W}$	w
	$\mathcal{R}$	wRw
	$\mathcal{I}$	$\langle\langle p,w\rangle,0\rangle$, $\langle\langle q,w\rangle,0\rangle$

In this model, $\mathcal{V}_{\mathfrak{M}}((p \vee q), w) = 0$. Since there is only one accessible world w from w, it follows that $\mathcal{V}_{\mathfrak{M}}(\Diamond(p \vee q), w) = 0$. Given this, it follows that the antecedent of 4 is true. However, $\mathcal{V}_{\mathfrak{M}}(p, w) = 0$ and $\mathcal{V}_{\mathfrak{M}}(\Diamond q), w) = 0$, so $\mathcal{V}_{\mathfrak{M}}((p \vee \Diamond q), w) = 0$. And, again, since only w is accessible from w, it follows that $\mathcal{V}_{\mathfrak{M}}\Box((p \vee \Diamond q), w) = 0$. So, the consequent of 4 is false and hence we have a countermodel.

Natural Deduction in PL: Basic Inference Rules

These are the basic inference rules for natural deduction in propositional logic (PL). We will add a set of *modal* inference rules below in order to extend the system to modal propositional logic (MPL).

In addition, we will add several derived rules to our inventory by proving these from our set of basic inference rules.

$$\frac{\begin{array}{c}(\phi \rightarrow \psi)\\ \phi\end{array}}{\psi}$$

modus ponens (**MP**)

$$\frac{\begin{array}{c}(\phi \rightarrow \psi)\\ \neg\psi\end{array}}{\neg\phi}$$

modus tollens (**MT**)

$$\frac{(\phi \wedge \psi)}{\begin{array}{c}\phi\\ \psi\end{array}}$$

simplication (**Simp**)

$$\frac{\begin{array}{c}\phi\\ \psi\end{array}}{(\phi \wedge \psi)} \qquad \frac{\begin{array}{c}\phi\\ \psi\end{array}}{(\psi \wedge \phi)}$$

addition (**Conj**)

$$\frac{\phi}{(\phi \vee \psi)} \qquad \frac{\phi}{(\psi \vee \phi)}$$

addition (**Add**)

$$\frac{\begin{array}{c}(\phi \vee \psi)\\ \neg\phi\end{array}}{\psi} \qquad \frac{\begin{array}{c}(\phi \vee \psi)\\ \neg\psi\end{array}}{\phi}$$

disjunctive syllogism (**DS**)

$$\frac{\neg\neg\phi}{\phi} \qquad \frac{\phi}{\neg\neg\phi}$$

double negation (**DN**)

$$\frac{\begin{array}{c}\phi \rightarrow \psi\\ \psi \rightarrow \phi\end{array}}{\phi \leftrightarrow \psi} \qquad \frac{\phi \leftrightarrow \psi}{\begin{array}{c}\phi \rightarrow \psi\\ \psi \rightarrow \phi\end{array}}$$

biconditional (**BC**)

$$\begin{array}{l|l}
\quad \phi & \text{ASS.} \\
\quad \vdots & \\
\quad \bot & \\
\neg\phi & \text{RAA}
\end{array}$$

Reductio ad Absurdum (**RAA**)

$$\begin{array}{l|l}
\quad \phi & \text{ASS.} \\
\quad \vdots & \\
\quad \psi & \\
\phi \rightarrow \psi & \text{CP}
\end{array}$$

Conditional Proof (**CP**)

Natural Deduction in PL: Exercises

- Prove the theorems below using the basic inference rules above.

Note: Once a theorem is proved, you may appeal to that theorem in derivations of other theorems, cf. info on *replacement rules* below.

1. $\vdash_{\text{PL}} \neg(\phi \vee \psi) \leftrightarrow (\neg\phi \wedge \neg\psi)$
2. $\vdash_{\text{PL}} \neg(\phi \wedge \psi) \leftrightarrow (\neg\phi \vee \neg\psi)$
3. $\vdash_{\text{PL}} (\phi \rightarrow \psi) \leftrightarrow (\neg\phi \vee \psi)$
4. $\vdash_{\text{PL}} \phi \leftrightarrow (\phi \wedge \phi)$
5. $\vdash_{\text{PL}} \phi \leftrightarrow (\phi \vee \phi)$
6. $\vdash_{\text{PL}} (\phi \rightarrow \psi) \leftrightarrow (\neg\psi \rightarrow \neg\phi)$
7. $\vdash_{\text{PL}} (\phi \wedge \psi) \leftrightarrow (\psi \wedge \phi)$
8. $\vdash_{\text{PL}} (\phi \vee \psi) \leftrightarrow (\psi \vee \phi)$
9. $\vdash_{\text{PL}} (\phi \wedge (\psi \wedge \chi)) \leftrightarrow ((\phi \wedge \psi) \wedge \chi)$
10. $\vdash_{\text{PL}} (\phi \vee (\psi \vee \chi)) \leftrightarrow ((\phi \vee \psi) \vee \chi)$
11. $\vdash_{\text{PL}} (\phi \wedge (\psi \vee \chi)) \leftrightarrow ((\phi \wedge \psi) \vee (\phi \wedge \chi))$
12. $\vdash_{\text{PL}} (\phi \vee (\psi \wedge \chi)) \leftrightarrow ((\phi \vee \psi) \wedge (\phi \vee \chi))$
13. $\vdash_{\text{PL}} ((\phi \wedge \psi) \rightarrow \chi) \leftrightarrow (\phi \rightarrow (\psi \rightarrow \chi))$
14. $\vdash_{\text{PL}} (\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\psi \rightarrow (\phi \rightarrow \chi))$

Solutions on the next pages...

Natural Deduction in PL: Solutions (Proofs of Replacement Rules)

1. $\vdash_{\text{PL}} \neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

Proof.

1	$\quad\mid\ \neg(p \vee q)$	assumption
2	$\quad\mid\quad\mid\ p$	assumption
3	$\quad\mid\quad\mid\ p \vee q$	Add., 2
4	$\quad\mid\quad\mid\ \bot$	1,3
5	$\quad\mid\ \neg p$	RAA, 2, 4
6	$\quad\mid\quad\mid\ q$	assumption
7	$\quad\mid\quad\mid\ p \vee q$	Add, 6
8	$\quad\mid\quad\mid\ \bot$	1,7
9	$\quad\mid\ \neg q$	RAA, 6, 8
10	$\quad\mid\ (\neg p \wedge \neg q)$	Conj., 5, 9
11	$\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$	CP, 1, 10
12	$\quad\mid\ \neg p \wedge \neg q$	assumption
13	$\quad\mid\quad\mid\ p \vee q$	assumption
14	$\quad\mid\quad\mid\ \neg p$	Simp., 12
15	$\quad\mid\quad\mid\ q$	DS, 13, 14
16	$\quad\mid\quad\mid\ \neg q$	Simp., 12
17	$\quad\mid\quad\mid\ \bot$	15,16
18	$\quad\mid\ \neg(p \vee q)$	RAA, 13, 17
19	$(\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$	CP, 12, 18
20	$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$	BC, 11, 19

2. $\vdash_{\text{PL}} \neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$

Proof.

1	$\quad \neg(p \wedge q)$	assumption
2	$\quad\quad \neg(\neg p \vee \neg q)$	assumption
3	$\quad\quad\quad \neg p$	assumption
4	$\quad\quad\quad \neg p \vee \neg q$	Add., 3
5	$\quad\quad\quad \bot$	2,4
6	$\quad\quad \neg\neg p$	RAA, 3, 5
7	$\quad\quad p$	DN, 6
8	$\quad\quad\quad \neg q$	assumption
9	$\quad\quad\quad \neg p \vee \neg q$	Add., 8
10	$\quad\quad\quad \bot$	2,9
11	$\quad\quad \neg\neg q$	RAA, 8, 10
12	$\quad\quad q$	DN, 11
13	$\quad\quad p \wedge q$	Conj., 7, 12
14	$\quad \neg(\neg p \vee \neg q) \rightarrow (p \wedge q)$	CP, 4, 13
15	$\quad \neg\neg(\neg p \vee \neg q)$	MT, 1, 14
16	$\quad (\neg p \vee \neg q)$	DN, 15
17	$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$	CP, 1, 16
18	$\quad \neg p \vee \neg q$	assumption
19	$\quad\quad p \wedge q$	assumption
20	$\quad\quad p$	Simp., 19
21	$\quad\quad \neg q$	DS, 19, 20
22	$\quad\quad q$	Simp., 19
23	$\quad\quad \bot$	21,22
24	$\quad \neg(p \wedge q)$	RAA, 19, 23
25	$(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$	CP, 19, 24
26	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$	BC, 17, 25

3. $\vdash_{\text{PL}} (p \to q) \leftrightarrow (\neg p \vee q)$

Proof.

1	$p \to q$	assumption
2	$\neg(\neg p \vee q)$	assumption
3	p	assumption
4	q	MP, 1, 3
5	$\neg p \vee q$	Add., 4
6	$\bot$	2,4
7	$\neg p$	RAA, 3, 6
8	$\neg p \vee q$	Add., 7
9	$\bot$	2,8
10	$\neg\neg(\neg p \vee q)$	RAA, 2, 9
11	$\neg p \vee q$	DN, 10
12	$(p \to q) \to (\neg p \vee q)$	CP, 1, 11
13	$\neg p \vee q$	assumption
14	p	assumption
15	q	DS, 13, 14
16	$p \to q$	CP, 14, 15
17	$(\neg p \vee q) \to (p \to q)$	CP, 13, 16
18	$(p \to q) \leftrightarrow (\neg p \vee q)$	BC, 12, 17

4. $\vdash_{PL} p \leftrightarrow (p \wedge p)$

Proof.

1	p	assumption
2	$\neg(p \wedge p)$	assumption
3	$\neg p \vee \neg p$	DeM, 2
4	$\neg p$	DS, 1, 3
5	$\bot$	1,4
6	$\neg\neg(p \wedge p)$	RAA, 2, 5
7	$(p \wedge p)$	DN, 6
8	$p \rightarrow (p \wedge p)$	CP, 1, 7
9	$p \wedge p$	assumption
10	p	Simp., 9
11	$(p \wedge p) \rightarrow p$	CP, 9, 10
12	$p \leftrightarrow (p \wedge p)$	BC, 8, 11

5. $\vdash_{PL} p \leftrightarrow (p \vee p)$

Proof.

1	p	assumption
2	$p \vee p$	Add., 1
3	$p \rightarrow (p \vee p)$	CP, 1, 2
4	$p \vee p$	assumption
5	$\neg p$	assumption
6	p	DS, 4, 5
7	$\bot$	5,6
8	$\neg\neg p$	RAA, 4, 7
9	p	DN, 8
10	$(p \vee p) \rightarrow p$	CP, 4, 9
11	$(p \vee p) \leftrightarrow p$	BC, 3, 10

6. $\vdash_{\text{PL}} (p \to q) \leftrightarrow (\neg q \to \neg p)$

Proof.

1	$\mid\ \mid\ p \to q$	assumption
2	$\mid\ \mid\ \mid\ \neg q$	assumption
3	$\mid\ \mid\ \mid\ \mid\ p$	assumption
4	$\mid\ \mid\ \mid\ \mid\ q$	MP, 1, 3
5	$\mid\ \mid\ \mid\ \mid\ \bot$	2,4
6	$\mid\ \mid\ \mid\ \neg p$	RAA, 3, 5
7	$\mid\ \mid\ \neg q \to \neg p$	CP, 2, 6
8	$\mid\ (p \to q) \to (\neg q \to \neg p)$	CP, 1, 7
9	$\mid\ \mid\ \neg q \to \neg p$	assumption
10	$\mid\ \mid\ \mid\ p$	assumption
11	$\mid\ \mid\ \mid\ \mid\ \neg q$	assumption
12	$\mid\ \mid\ \mid\ \mid\ \neg p$	MP, 9, 11
13	$\mid\ \mid\ \mid\ \mid\ \bot$	10,12
14	$\mid\ \mid\ \mid\ \neg\neg q$	RAA, 11, 13
15	$\mid\ \mid\ \mid\ q$	DN, 14
16	$\mid\ \mid\ p \to q$	CP, 10, 15
17	$\mid\ (\neg q \to \neg p) \to (p \to q)$	CP, 9, 16
18	$\mid\ (p \to q) \leftrightarrow (\neg q \to \neg p)$	BC, 8, 17

7. $\vdash_{PL} (p \wedge q) \to (q \wedge p)$

Note: We only prove the left-to-right directions here since the proving the opposite direction becomes trivial.

Proof.

1	$\quad p \wedge q$	assumption
2	$\quad p$	Simp., 1
3	$\quad q$	Simp., 1
4	$\quad q \wedge p$	Conj., 2, 3
5	$(p \wedge q) \to (q \wedge q)$	CP, 1, 4

8. $\vdash_{PL} (p \vee q) \to (q \vee p)$

Proof.

1	$\quad p \vee q$	assumption
2	$\quad\quad \neg(q \vee p)$	assumption
3	$\quad\quad \neg q \wedge \neg p$	DeM., 2
4	$\quad\quad \neg q$	Simp., 3
5	$\quad\quad p$	DS, 1, 4
6	$\quad\quad \neg p$	Simp., 3
7	$\quad\quad \bot$	5,6
8	$\quad \neg\neg(q \vee p)$	RAA, 2, 7
9	$\quad q \vee p$	DN, 8
10	$(p \vee q) \to (q \vee p)$	CP, 1, 9

9. $\vdash_{PL} (\phi \wedge (\psi \wedge \chi)) \leftrightarrow ((\phi \wedge \psi) \wedge \chi)$

Proof.

10. $\vdash_{PL} (\phi \vee (\psi \vee \chi)) \leftrightarrow ((\phi \vee \psi) \vee \chi)$

Proof.

11. $\vdash_{PL} (\phi \wedge (\psi \vee \chi)) \leftrightarrow ((\phi \wedge \psi) \vee (\phi \wedge \chi))$

Proof.

12. $\vdash_{\text{PL}} (\phi \vee (\psi \wedge \chi)) \leftrightarrow ((\phi \vee \psi) \wedge (\phi \vee \chi))$

Proof.

13. $\vdash_{\text{PL}} ((\phi \wedge \psi) \to \chi) \leftrightarrow (\phi \to (\psi \to \chi))$

Proof.

14. $\vdash_{\text{PL}} (\phi \to (\psi \to \chi)) \leftrightarrow (\psi \to (\phi \to \chi))$

Proof.

Replacement Rules

- For each theorem above, we add a corresponding *replacement rule*. A replacement rule, '$\phi :: \psi$', is a rule that allows the replacement of ϕ for ψ any point in a proof and vice versa (citing the relevant rule).

1.	$(\neg\phi \vee \neg\psi) :: \neg(\phi \wedge \psi)$	DeMorgan	(DeM)
2.	$(\neg\phi \wedge \neg\psi) :: \neg(\phi \vee \psi)$	DeMorgan	(DeM)
3.	$(\phi \to \psi) :: (\neg\phi \vee \psi)$	Material Implication	(IMP)
3.	$(\phi \to \psi) :: (\neg\phi \vee \psi)$	Material Implication	(IMP)
4.	$(\phi \vee \phi) :: \phi$	Idempotence	(Idem)
5.	$(\phi \wedge \phi) :: \phi$	Idempotence	(Idem)
6.	$(\phi \to \psi) :: (\neg\psi \to \neg\phi)$	Contraposition	(Cont)
7.	$(\phi \wedge \psi) :: (\psi \wedge \phi)$	Commutativity	(Com)
8.	$(\phi \vee \psi) :: (\psi \vee \phi)$	Commutativity	(Com)
9.	$(\phi \wedge (\psi \wedge \chi)) :: (\phi \wedge \psi) \wedge \chi)$	Associativity	(Assoc)
10.	$(\phi \vee (\psi \vee \chi)) :: (\phi \vee \psi) \vee \chi)$	Associativity	(Assoc)
11.	$(\phi \wedge (\psi \vee \chi)) :: ((\phi \wedge \psi) \vee (\phi \wedge \chi))$	distribution	(Dist)
12.	$(\phi \vee (\psi \wedge \chi)) :: ((\phi \vee \psi) \wedge (\phi \vee \chi))$	distribution	(Dist)
13.	$((\phi \wedge \psi) \to \chi) :: (\phi \to (\psi \to \chi))$	exportation	(Exp)
14.	$(\phi \to (\psi \to \chi)) :: (\psi \to (\phi \to \chi))$	permutation	(Per)

Natural Deduction in MPL: Basic Modal Inference Rules

Note: These rules only cover the modal logics T, S4, and S5.

$$\frac{\neg\Diamond\neg\phi}{\Box\phi} \qquad \frac{\Box\phi}{\neg\Diamond\neg\phi}$$

modal negation (**MN**)

$$\frac{\neg\Box\neg\phi}{\Diamond\phi} \qquad \frac{\Diamond\phi}{\neg\Box\neg\phi}$$

modal negation (**MN**)

$$\frac{\neg\Box\phi}{\Diamond\neg\phi} \qquad \frac{\neg\Diamond\phi}{\Box\neg\phi}$$

modal negation (**MN**)

$$\frac{\Box\neg\phi}{\neg\Diamond\phi} \qquad \frac{\Diamond\neg\phi}{\neg\Box\phi}$$

modal negation (**MN**)

$$\frac{\phi}{\Diamond\phi}$$

possibility introduction (**PI**)

$$\frac{\Box\phi}{\phi}$$

modal reiteration T (**MRT**)

$$\frac{\Box\phi}{\Box\phi}$$

modal reiteration S4 (**MRS4**)

$$\frac{\Diamond\phi}{\Diamond\phi}$$

modal reiteration S5 (**MRS5**)

| |	NULL ASS.
| |—	
| | ⋮	
| | ϕ	
| $\Box\phi$	NI

Necessity Introduction (**NI**)

Rules for null assumption proofs:

1. There is *no assumption* (or a null assumption).
2. Every formula inside the scope of a null assumption must be deduced from preceding formulas *inside the scope of* that null assumption unless it is deduced by modal reiteration for the appropriate modal logic.
3. The proof ends with a discharge of the null assumption line. After the discharge of the null assumption line where the formula immediately before the discharge is ϕ, the next formula is $\Box\phi$.
4. A null assumption proof may contain a subproof, but anything derived via a subproof must rely on no assumptions.

Natural Deduction in MPL: Exercises

- Prove the theorems below using the *basic inference rules* and the *basic modal inference rules* above.

1. $\vdash_{S4} \Box p \to \Box\Box\Box p$
2. $\vdash_{T} \neg\Box p \vee (p \vee q)$
3. $\vdash_{S4} \Box p \to \Box\Box\Diamond p$
4. $\vdash_{T} \Box\big((p \to q) \to (p \to p)\big)$
5. $\vdash_{T} \Box(p \to (p \vee q))$
6. $\vdash_{T} \Box(p \to q) \to (\Box p \to \Box q)$
7. $\Diamond\Diamond p \vdash_{S4} \Diamond p$
8. $\vdash_{T} (\Box p \wedge \neg\Diamond q) \to \neg(\neg p \vee \Box q)$
9. $\Box(p \vee q), \Box(p \to r), \Box(q \to r) \vdash_{T} \Box r$
10. $\vdash_{T} \Diamond\neg\neg p \to \Diamond p$

Solutions on the next pages...

Natural Deduction in MPL: Solutions

1. $\vdash_{S4} \Box p \to \Box\Box\Box p$

1	$\quad \mid \Box p$	assumption
2	$\quad \mid \quad \mid$	null assumption
3	$\quad \mid \quad \mid \Box p$	MRS4, 1
4	$\quad \mid \Box\Box p$	NI, 3
5	$\quad \mid \quad \mid$	null assumption
6	$\quad \mid \quad \mid \Box\Box p$	MRS4, 3
7	$\quad \mid \Box\Box\Box p$	NI, 6
8	$\Box p \to \Box\Box\Box p$	CP, 1, 8

2. $\vdash_{T} \neg\Box p \vee (p \vee q)$

1	$\quad \mid \Box p$	assumption
2	$\quad \mid p$	MRT, 1
3	$\quad \mid p \vee q$	Add, 2
4	$\Box p \to (p \vee q)$	CP, 1, 3
5	$\neg\Box p \vee (p \vee q)$	IMP, 1, 3

3. $\vdash_{S4} \Box p \to \Box\Box\Diamond p$

1	$\Box p$	assumption
2		null assumption
3	$\Box p$	MRS4, 1
4	p	MRT, 3
5	$\Diamond p$	PI, 3
6	$\Box\Diamond p$	NI, 5
7		null assumption
8	$\Box\Diamond p$	MRS4, 7
9	$\Box\Box\Diamond p$	NI, 8
10	$\Box p \to \Box\Box\Diamond p$	CP, 1, 10

4. $\vdash_{T} \Box\big((p \to q) \to (p \to p)\big)$

1		null assumption
2	$p \to q$	assumption
3	p	assumption
4	$p \wedge p$	Idem, 3
5	p	Simp, 4
6	$p \to p$	CP, 3, 5
7	$(p \to q) \to (p \to p)$	CP, 2, 6
8	$\Box\big((p \to q) \to (p \to p)\big)$	NI, 7

5. $\vdash_{T} \Box(p \to (p \vee q))$

1		**null assumption**
2	p	assumption
3	$p \vee q$	Add, 2
4	$p \to (p \vee q)$	CP, 2–3
5	$\Box(p \to (p \to q))$	NI, 1–4

6. $\vdash_{T} \Box(p \to q) \to (\Box p \to \Box q)$

1	$\mid\ \mid\ \Box(p \to q)$	ass.
2	$\mid\ \mid\ \mid\ \Box p$	ass.
3	$\mid\ \mid\ \mid\ \mid$	**null assumption**
4	$\mid\ \mid\ \mid\ \mid\ (p \to q)$	MRT, 1
5	$\mid\ \mid\ \mid\ \mid\ p$	MRT, 2
6	$\mid\ \mid\ \mid\ \mid\ q$	MP, 4, 5
7	$\mid\ \mid\ \mid\ \Box q$	NI, 3–6
8	$\mid\ \mid\ \Box p \to \Box q$	CP, 2, 8
9	$\mid\ \Box(p \to q) \to (\Box p \to \Box q)$	CP, 1–8

7. $\Diamond\Diamond p \vdash_{S4} \Diamond p$

1	$\mid\ \Diamond\Diamond p$	premise
2	$\mid\ \mid\ \Box\neg p$	ass.
3	$\mid\ \mid\ \mid$	**null assumption**
4	$\mid\ \mid\ \mid\ \Box\neg p$	MRS4, 2
5	$\mid\ \mid\ \Box\Box\neg p$	NI, 2–4
6	$\mid\ \Box\neg p \to \Box\Box\neg p$	CP, 2, 5
7	$\mid\ \neg\Box\Box\neg p \to \neg\Box\neg p$	Cont., 6
8	$\mid\ \Diamond\neg\Box\neg p \to \neg\Box\neg p$	MN, 7
9	$\mid\ \Diamond\Diamond p \to \neg\Box\neg p$	MN, 8
10	$\mid\ \Diamond\Diamond p \to \Diamond p$	MN, 9
11	$\mid\ \Diamond p$	MP, 1, 10

8. $\vdash_T (\Box p \wedge \neg\Diamond q) \rightarrow \neg(\neg p \vee \Box q)$

1	$\Box p \wedge \neg\Diamond q$	ass.
2	$\Box p$	Simp., 1
3	$\neg\Diamond q$	Simp., 1
4	$\Box\neg q$	MN, 3
5	$\neg q$	MRT, 4
6	$\Diamond\neg q$	PI, 5
7	$\neg\Box q$	MN, 6
8	p	MRT, 2
9	$p \wedge \neg\Box q$	Conj, 7, 8
10	$\neg(\neg p \vee \neg\neg\Box q)$	DeM, 9
11	$\neg(\neg p \vee \Box q)$	DN, 10
12	$(\Box p \wedge \neg\Diamond q) \rightarrow \neg(\neg p \vee \Box q)$	CP, 1, 11

9. $\Box(p \vee q), \Box(p \to r), \Box(q \to r) \vdash_T \Box r$

1	$\Box(p \vee q)$	premise
2	$\Box(p \to r)$	premise
3	$\Box(q \to r)$	premise
4		null assumption
5	$p \vee q$	MRT, 1
6	$p \to r$	MRT, 2
7	$q \to r$	MRT, 3
8	$\neg r$	assumption
9	$\neg p$	MT, 6, 8
10	q	DS, 5, 9
11	r	MP, 7, 10
12	$\bot$	8, 11
13	$\neg\neg r$	RAA, 8–12
14	r	DN, 13
15	$\Box r$	NI, 4–13

10. $\vdash_T \Diamond\neg\neg p \to \Diamond p$

Proof without illicit use of DN within complex formulas.

This includes a proof of the theorem: $\Box(p \to q) \to (\Diamond p \to \Diamond q)$

1		null assumption
2	$\neg\neg p$	assumption
3	p	DN, 2
4	$\neg\neg p \to p$	CP, 2, 3
5	$\Box(\neg\neg p \to p)$	NI, 1–4
6	$\Diamond\neg\neg p$	assumption
7	$\neg\Box\neg\neg\neg p$	def. $\Diamond$, 6
8	$\Box\neg p$	assumption
9		null assumption
10	$\neg\neg p \to p$	MRT, 5
11	$\neg p$	MRT, 8
12	$\neg\neg\neg p$	MT, 10, 11
13	$\Box\neg\neg\neg p$	NI, 9–12
14	$\Box\neg p \to \Box\neg\neg\neg p$	CP, 8, 13
15	$\neg\Box\neg p$	MT, 7, 14
16	$\Diamond p$	def. $\Diamond$, 15
17	$\Diamond\neg\neg p \to \Diamond p$	CP, 6, 16

11. $\vdash_T (\Diamond p \vee \Diamond q) \rightarrow \Diamond(p \vee q)$

1	$\Diamond p$	assumption
2	$\neg\Diamond(p \vee q)$	assumption
3	$\Box\neg(p \vee q)$	MN, 2
4		null assumption
5	$\neg(p \vee q)$	MRT, 3
6	$\neg p \wedge \neg q$	DeM, 5
7	$\neg p$	Simp., 6
8	$\Box\neg p$	NI, 4–7
9	$\neg\Diamond p$	MN, 8
10	$\bot$	1, 9
11	$\neg\neg\Diamond(p \vee q)$	RAA, 2, 10
12	$\Diamond(p \vee q)$	DN, 11
13	$\Diamond p \rightarrow \Diamond(p \vee q)$	CP, 1, 12
14	$\Diamond q$	assumption
15	$\neg\Diamond(p \vee q)$	assumption
16	$\Box\neg(p \vee q)$	MN, 15
17		null assumption
18	$\neg(p \vee q)$	MRT, 16
19	$\neg p \wedge \neg q$	DeM, 18
20	$\neg q$	Simp., 19
21	$\Box\neg q$	NI, 17–20
22	$\neg\Diamond q$	MN, 21
23	$\bot$	14, 22
24	$\neg\neg\Diamond(p \vee q)$	RAA, 15, 23
25	$\Diamond(p \vee q)$	DN, 24
26	$\Diamond q \rightarrow \Diamond(p \vee q)$	CP, 14, 25

27	$\neg\Diamond(p \vee q)$	assumption
28	$\neg\Diamond p$	MT, 13, 27
29	$\neg\Diamond q$	MT, 26, 27
30	$\neg\Diamond p \wedge \neg\Diamond q$	Conj, 28, 29
31	$\neg(\Diamond p \vee \Diamond q)$	DeM, 30
32	$\neg\Diamond(p \vee q) \rightarrow \neg(\Diamond p \vee \Diamond q)$	CP, 27, 31
33	$(\Diamond p \vee \Diamond q) \rightarrow \Diamond(p \vee q)$	Cont, 32